MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1937 JANUARY 8

Mr. J. H. Reynolds, President, in the Chair.

The President announced that His Majesty King George VI had been graciously pleased to become the Patron of the Society.

The President further announced that a stand to view the Coronation Procession on May 12 would be erected in front of Burlington House for the joint use of the Societies occupying the building, and that a limited number of seats would be available for the Royal Astronomical Society. These would be allotted by ballot, and a circular was being sent to all Fellows immediately.

The election by the Council of the following Fellows was duly confirmed :—

M. Akhtar, M.Sc., Imperial College of Science, S. Kensington, London, S.W. 7 ;

G. L. Clark, B.A., 164 Gilbert Road, Cambridge ;

Mrs. Rosina Dafter, Flower Road, Northgate, Brisbane, Queensland, Australia ;

William Samuel Davenport, Commander, R.D., R.N.R., F.R.Met.Soc., Questings, Arundel Road, Peacehaven, Sussex ;

J. Fleming, B.A., Trinity Hall, Cambridge ;

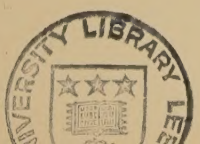
Sydney I. Luck, 26 Wellington Square, Chelsea, S.W. 3 ;

Eugene Fairfield MacPike, 135 East Eleventh Place, Chicago, Ill., U.S.A. ;

Kenneth Mitchell, Ph.D., B.A., Assistant Lecturer in Mathematics, The University, Leeds, 2 ;

Charles Mowat, Leith Nautical College, Edinburgh, 6 ;

Thornton Leigh Page, University Observatory, Oxford ;



John Fraser Paterson, Box 151A, Broken Hill, N.S.W., Australia ;
C. F. Retter, M.A., B.Sc., 90 Blake Road, New Southgate, London, N. 11 ;
The Dowager Lady Saltoun, 25 Gloucester Place, London, W. 1 ;
George Whalley Wakeford, Lieutenant, R.N.R., Dept. of Navigation, University College, Southampton ; and
Oswald Martin Watts, A.I.N.A., 286 Leigham Court Road, Streatham, London, S.E. 16.

The following candidate was proposed for election as a Fellow of the Society, the name of the proposer from personal knowledge being appended :—

Colin Kerr Grant, 2 Brookside, Cambridge (proposed by Sir G. P. Lenox-Conyngham).

Seventy-three presents were announced as having been received since the last meeting.

The President announced that the Gold Medal of the Society had been awarded to Dr. Harold Jeffreys for his researches into the physics of the Earth and other planets, and for his contributions to the study of the origin and age of the Solar System.

THE EFFECT OF RED-SHIFT ON THE MAGNITUDES OF NEBULÆ.

Sir A. S. Eddington, F.R.S.

1. As the result of an extensive survey of the distribution of faint extra-galactic nebulæ, coupled with a careful investigation of the possible sources of error, Dr. Hubble * has concluded that, between $18^{m}.5$ and $21^{m}.0$, the number N of nebulæ brighter than photographic magnitude m is represented by

$$\log_{10} N = 0{\cdot}501m + \text{const.} \qquad (1)$$

within the uncertainties of the data. It is well known that for a uniform distribution, provided that there is no weakening of the light other than the ordinary decrease with the inverse-square of the distance, N must satisfy

$$\log_{10} N = 0{\cdot}6m + \text{const.} \qquad (2)$$

If the actual distribution is uniform, the difference between the coefficients

* *Ap. J.*, 84, 517, 1936. It is also necessary to refer to an earlier paper, *ibid.*, p. 270.

in (1) and (2) may be attributed to the red-shift. On this basis Hubble has made a quantitative determination of the effect of red-shift on magnitude. His result diverges from the calculated effect of a velocity of recession to an extent which he believes to be significant, and appears to support the view that the red-shift arises from some cause other than recession.

If Δm is the change of magnitude due to the red-shift, we shall call $m - \Delta m$ the "corrected magnitude." For a uniform distribution statistics of corrected magnitudes should satisfy the relation (2). Using a mean correction $\overline{\Delta m}$ for the nebulæ of magnitude m, Hubble accordingly sets

$$\log N = 0{\cdot}6(m - \overline{\Delta m}) + \text{const.} \qquad (3)$$

This would be strictly true if the nebulæ of magnitude m were all at the same distance and had therefore the same red-shift correction ; but the dispersion in distance is large, and the corresponding dispersion of Δm will scatter the nebulæ of magnitude m through a considerable range of corrected magnitudes. It is not at all obvious that this will have the same effect on the statistics as the application of a mean correction $\overline{\Delta m}$.

The purpose of the present investigation is to replace this questionable step by a rigorous treatment which takes account of the dispersion of the absolute magnitudes of the nebulæ. The solution of what appeared to be a rather intractable problem has turned out to be unexpectedly simple. From an observed frequency distribution $N(m)$ it is possible to deduce the true frequency $N'(m)$ corrected for the effects of accidental error of observation of m. By a mathematically identical process we remove the dispersion of absolute magnitude, and deduce from the observed statistics the function $N'(m)$ representing the counts which would have been obtained if the nebulæ had all had the same absolute magnitude (red-shifts, etc., being unaltered). Hubble's procedure can then be legitimately applied to N'.

The consequence of taking account of the dispersion (for which Hubble's determination $\sigma = 0^{m}{\cdot}85$ is adopted) is that his result $\Delta m = 2{\cdot}94 d\lambda/\lambda$ is changed to $\Delta m = 3{\cdot}06 d\lambda/\lambda$. Thus the effect of the dispersion turns out to be unimportant.*

2. Let $u(m) = dN/dm$, so that the number of nebulæ between magnitudes m and $m + dm$ is $u(m)dm$. Following Hubble, we assume that this corresponds to a uniform distribution in space, and that in any volume the proportion with absolute magnitudes between M and $M + dM$ is given by the Gaussian law

$$\frac{h}{\sqrt{\pi}} e^{-h^2(M - M_0)^2} dM. \qquad (4)$$

For comparison we consider also a uniform distribution of nebulæ all of the same absolute magnitude M_0. Let the corresponding number with magnitudes between m and $m + dm$ be $v(m)dm$. Both $u(m)$ and $v(m)$ refer to the uncorrected magnitudes affected by red-shift. We shall distinguish

* This is contrary to the conclusion originally announced. An algebraic mistake in computing the derivatives of N' was found in proof-reading, and the last part of the paper after equation (20) has been modified.

the systems which furnish $u(m)$ and $v(m)$ as the "actual system" and the "ideal system."

To a nebula of magnitude m' in the ideal system there corresponds a nebula of magnitude $m = m' + (M - M_0)$ in the actual system. The differences $m - m'$, being equal to $M - M_0$, are distributed according to the Gaussian law and are uncorrelated with m'. Thus the distribution of m is the same as that which would be given by introducing an accidental error in the measurement of m', the error ϵ having the Gaussian law

$$\frac{h}{\sqrt{\pi}} e^{-h^2\epsilon^2} d\epsilon. \tag{5}$$

Conversely $v(m)$ is derived from $u(m)$ by correcting for the effects of such an error.

The relation between $v(m)$ and $u(m)$ is accordingly *

$$v(m) = e^{-D^2/4h^2} u(m), \tag{6}$$

where $D = d/dm$.

To put the argument in another form : since Dr. Hubble in his discussion treats the observed magnitudes as though they were magnitudes of a standard source of light, we must for consistency treat the differences of the actual sources from the standard source (reproduced directly in the observed magnitudes) as errors of observation ; and (before applying his method) we must correct the observed frequency curve for this observational error.

3. Let us first apply (6) to the distribution specified by (1). Since N is then of the form Ae^{bm}, we have

$$u(m) = dN/dm = bAe^{bm}.$$

When operating on the form e^{bm}, D reduces to an eigen value b, and

$$e^{-D^2/4h^2} = e^{-b^2/4h^2}. \tag{7}$$

Thus $v(m)$ differs from $u(m)$ only by the constant factor $e^{-b^2/4h^2}$, which does not concern us ; and N in (1) may be taken to refer indifferently to the actual or to the ideal system.

At first sight this seems to validate Hubble's assumption. He has not explicitly taken into account the dispersion in absolute magnitude ; but this is unnecessary because it happens that the statistics apply unchanged to an ideal system in which the nebulæ have all the same absolute magnitude M_0, and we may suppose that the ideal system is being discussed. All the nebulæ of magnitude m will then have the same distance r given by †

$$\log_{10} r = 0{\cdot}2(m - \Delta m - M_0). \tag{8}$$

Hubble's method of determining Δm from the statistics applies rigorously, and for each magnitude m gives the correction Δm corresponding to the redshift $d\lambda/\lambda$ for the above distance r.

* Eddington, *M.N.*, **73**, 359, 1913.

† The unit of distance is 10 parsecs.

But the confirmation is only apparent ; for Hubble has naturally assumed that his correction Δm corresponds to the red-shift for the distance r' given by

$$\log_{10} r' = 0{\cdot}2(m - \Delta m - M_0'), \tag{9}$$

where M_0' is the mean absolute magnitude of the nebulæ of magnitude m. M_0' is nearly a magnitude brighter than M_0, which is the mean absolute magnitude of the nebulæ in a unit volume.

4. If Δm is neglected, the distance of a nebula of magnitude m and absolute magnitude M is

$$r = e^{0{\cdot}461(m-M)}.$$

Keeping M constant, the volume element

$$4\pi r^2 dr = 4\pi r^3 d(\log r) = 4\pi(0{\cdot}461)e^{1{\cdot}382(m-M)}dm. \tag{10}$$

To obtain the frequency of absolute magnitudes of nebulæ of apparent magnitude m, we must weight the frequency in unit volume given by (4) proportionately to (10). The frequency is therefore proportional to

$$e^{-h^2(M-M_0)^2 - 1{\cdot}382M}$$

or to

$$e^{-h^2(M-M_0')^2},$$

where

$$2h^2M_0 - 1{\cdot}382 = 2h^2M_0'. \tag{11}$$

The dispersion $\sigma = 1/h\sqrt{2}$. Thus (11) gives

$$M_0' = M_0 - 1{\cdot}382\sigma^2. \tag{11a}$$

Using Hubble's determination $\sigma = 0{\cdot}85$, we have $M_0' = M_0 - 1{\cdot}00$. Hence by (8) and (9)

$$r' = 1{\cdot}585r.$$

If Δm is not neglected, M_0' will vary with m. But we are here only concerned with the range of m used by Hubble in his determination of the dependence of $d\lambda/\lambda$ on m. As this did not embrace such faint nebulæ as those used in the statistical discussions, the effect of Δm is insignificant.

Through using r' instead of r, Hubble has associated Δm with a red-shift which is too large in the ratio 1·585. His result $\Delta m = 2{\cdot}94d\lambda/\lambda$ must therefore be amended to

$$\Delta m = 4{\cdot}66d\lambda/\lambda. \tag{12}$$

This preliminary result requires a correction which will be determined in §§ 5, 6.

5. The empirical formula (1) cannot be trusted beyond the range of $2\frac{1}{2}$ magnitudes from which it was derived. Hubble, in fact, does not utilise it as it stands ; he replaces it by a formula whose form is prescribed by theory, but containing disposable constants adjusted to fit the observed values of N between $18^{m}{\cdot}5$ and $21^{m}{\cdot}0$. For a rigorous treatment we must proceed in the same way.

Following Hubble, we set $\Delta m = kd\lambda/\lambda$, the observational determination of k being the chief aim of the investigation. Since $d\lambda/\lambda$ is proportional to r, we have

$$d\lambda/\lambda = Ae^{0\cdot461(m-\Delta-M)}, \tag{13}$$

$$\Delta = kAe^{0\cdot461(m-\Delta-M)}, \tag{14}$$

where A is constant. (We use Δ as an abbreviation for Δm, the double symbol being somewhat confusing to work with.) In the ideal system ($M = M_0$), (14) gives

$$\Delta = Ce^{\alpha(m-\Delta)} \tag{15}$$

where

$$\alpha = 0\cdot461, \qquad C = kAe^{-\alpha M_0}. \tag{16}$$

Then by the formula (2) for uniform distribution,

$$\log_e N' = 3\alpha(m-\Delta) + \text{const.} \tag{17}$$

We use N' for the number of nebulæ brighter than m in the ideal system to distinguish it from the corresponding number N in the actual system.

Since $v(m) = dN'/dm$, $u(m) = dN/dm$, the relation $u(m) = e^{D^2/4\lambda^2}v(m)$ leads to

$$N = e^{D^2/4\lambda^2}N' = e^{\frac{1}{2}\sigma^2D^2}N'. \tag{18}$$

There is no constant of integration, since N and N' must vanish together when m becomes small. We proceed to evaluate $e^{\frac{1}{2}\sigma^2D^2}$ for the operand N'. From (15) and (17) we obtain

$$N' = p\Delta^3 \qquad (p = \text{const.}). \tag{19}$$

Also from (15)

$$\frac{d\Delta}{dm} = \alpha\Delta\left(1 - \frac{d\Delta}{dm}\right),$$

so that

$$\frac{d\Delta}{dm} = \frac{\alpha\Delta}{1+\alpha\Delta}. \tag{20}$$

From (19) and (20) we now easily obtain the successive derivatives of N'. The even derivatives required in (18) are:

$$\begin{aligned}
d^2N'/dm^2 &= N'\alpha^2(9+6\alpha\Delta)(1+\alpha\Delta)^{-3},\\
d^4N'/dm^4 &= N'\alpha^4(81+42\alpha\Delta+6\alpha^2\Delta^2)(1+\alpha\Delta)^{-7},\\
d^6N'/dm^6 &= N'\alpha^6(729-2082\alpha\Delta+300\alpha^2\Delta^2+312\alpha^3\Delta^3+36\alpha^4\Delta^4)(1+\alpha\Delta)^{-11},\\
d^8N'/dm^8 &= N'\alpha^8(6561-78510\alpha\Delta+207810\alpha^2\Delta^2-128160\alpha^3\Delta^3-9660\alpha^4\Delta^4\\
&\qquad +6696\alpha^5\Delta^5+720\alpha^6\Delta^6)(1+\alpha\Delta)^{-15}.
\end{aligned} \tag{21}$$

In the computations in § 6 the term involving the eighth derivative was sometimes just appreciable. Inserting these values in (18), we obtain N/N' as a function of Δ and σ, say $f(\Delta, \sigma)$.

Hence by (17)

$$\log_{10} N = 0\cdot6(m-\Delta) + \log_{10} f(\Delta, \sigma) + \text{const.} \tag{22}$$

This formula is to be used in conjunction with (15), viz.

$$\Delta = Ce^{\alpha(m-\Delta)}. \tag{23}$$

6. From (22) and (23) we have to determine the value of C which best fits the observed values of N. This is done by successive approximations as in Hubble's discussion, which employs (22) without the term containing σ.

To complete the determination of k we must find A from the radial velocities of nebulæ of known magnitude. By (13)

$$\log(d\lambda/\lambda) = \alpha(m - \Delta - M) + \log A.$$

Taking the mean for nebulæ of magnitude m,

$$\overline{\log(d\lambda/\lambda)} = \alpha(m - \overline{\Delta} - M_0') + \log A. \tag{24}$$

The correction $\overline{\Delta}$ for the nebulæ bright enough to be measured spectroscopically is unimportant. We can therefore obtain from the observational data $(-\alpha M_0' + \log A)$, or its antilogarithm $Ae^{-\alpha M_0'}$. Since we also know $kAe^{-\alpha M_0}(=C)$, and $M_0 = M_0' + 1{\cdot}00$ by § 4, we can deduce the value of k.

To avoid a complete rediscussion, we shall derive corrections to Hubble's results. Hubble has found

$$\log_{10}\Delta m = 0{\cdot}2(m - \Delta m) - 4{\cdot}239. \tag{25}$$

This was obtained from the nebular counts by a procedure here superseded. We are not, however, concerned with its derivation, but with the result subsequently found that in order that (25) may agree with the radial velocity determinations k must be 2·94. That is to say, (25) will represent Δm for a nebula of absolute magnitude M_0' if $k = 2{\cdot}94$, or of absolute magnitude M_0 (as in the ideal system) if $k = 4{\cdot}66$. From (25) we calculate for the ideal system

m	$\Delta m(k=4{\cdot}66)$	$\Delta m(k=3{\cdot}1)$
18·5	·257	·177
19·75	·423	·298
21·0	·671	·486

A preliminary solution shows that the value of k is about 3·1 ; we have therefore added the third column. As an example of the calculation of the third column, consider a star of corrected magnitude $m - \Delta m = 20{\cdot}514$. By (25) we obtain $\Delta m = {\cdot}731$; but this is to be reduced in the ratio $3{\cdot}1/4{\cdot}66$, giving $\Delta m = {\cdot}486$, and the corresponding apparent magnitude is $20{\cdot}514 + {\cdot}486 = 21{\cdot}0$. Using the third column, with $\sigma = 0{\cdot}85$ as given by Hubble, we find by (18)

$$\log_{10}(N/N') = {\cdot}2398,\ {\cdot}2105,\ {\cdot}1759 \quad \text{for} \quad 18^{\text{m}}{\cdot}5,\ 19^{\text{m}}{\cdot}75,\ 21^{\text{m}}{\cdot}0.$$

Thus in the interval surveyed the gradient of $\log_{10}(N'/N)$ is ·0255 per magnitude. By (1) the gradient of $\log_{10} N$ is ·501 ; hence the gradient of $\log_{10} N'$ is ·5265.

As has been explained, our method consists in deducing from the observed statistics the statistics of an ideal system and working with the latter. In §§ 3, 4 we found no change in the statistics in passing from the actual to

the ideal system; but the results were provisional since they involved improper extrapolation of an empirical formula. Using an extrapolatable formula we now find that the gradient is increased by ·0255 in the ideal system.

The gradient for corrected magnitudes is 0·6; thus the gradient ·5265 for apparent magnitudes signifies that the range $2^{m}\cdot5$ of apparent magnitudes corresponds to a range $2^{m}\cdot194$ of corrected magnitudes; that is to say, the change $\delta(\Delta m)$ of Δm between $18^{m}\cdot5$ and $21^{m}\cdot0$ must be ·306. From the table given above $\delta(\Delta m)$ is ·414 if $k=4\cdot66$, and ·309 if $k=3\cdot1$. By trial we find that $\delta(\Delta m)=\cdot306$ corresponds to $k=3\cdot06$; so that

$$\Delta m = 3\cdot06 d\lambda/\lambda. \qquad (26)$$

We also verify in the same way that the gradient corresponding to $k=4\cdot66$ is ·501; this verification shows that the change from 2·94 to 3·06 represents the effect of dispersion alone, the two determinations being equivalent in all other respects.

7. Since the investigation has failed to alter k appreciably, it may be permissible to add a short discussion of the significance of the value found. Examples of the relation between k and $\delta(\Delta m)$ have been given above; the following table shows the relation more systematically:—

$k=$	2·5	3·0	3·5	4·0	4·5
$\delta(\Delta m)=$	·262	·301	·338	·372	·404

The quantity determined by applying the hypothesis of uniform distribution to the nebular counts is $\delta(m-\Delta m)$; since the difference δm of limiting magnitude of the two counts is determined photometrically, this yields $\delta(\Delta m)$, which is transformed into a determination of k by the above table. An error of $0^{m}\cdot01$ in δm will produce an equal error in $\delta(\Delta m)$ and an error of ·13 in k; in particular the discordance of our determination of k from the value 4·0 expected (according to the usual interpretation of the red-shift as a Doppler effect) is equivalent to a discordance of $0^{m}\cdot066$ in δm.

Hubble finds the probable accidental error of m for the two extreme counts to be ± ·03. This corresponds to ± ·042 in δm, and ± ·56 in k. (For the intermediate counts m is given only to one decimal place, so that they carry no weight in a determination requiring great delicacy.) Thus taking into account this one source of error alone, the result should be expressed as $k=3\cdot1\pm0\cdot6$.

Conclusion

On the hypothesis of uniform distribution, Hubble has deduced from counts of faint nebulæ that the effect of red-shift on magnitude is $\Delta m = 2\cdot94 d\lambda/\lambda$. The theoretical effect is $\Delta m = 4\cdot0 d\lambda/\lambda$ if the red-shift is due to recession, or $3\cdot0 d\lambda/\lambda$ if it is due to other suggested causes. In deriving the coefficient 2·94, the dispersion of the absolute magnitudes of the nebulæ was neglected. Formulæ are here developed which take account of the dispersion. By including the dispersion the observational result is changed to $\Delta m = 3\cdot06 d\lambda/\lambda$. The correction thus proves to be unimportant.

It is pointed out, however, that (taking into account only one recognised source of accidental error) the probable error of k is 0·56; so that the difference from the expected value 4·0 is scarcely sufficient to constitute a discordance.

I am indebted to Dr. Hubble, who kindly sent me an advance proof of his paper desiring scrutiny of the theoretical treatment.

Observatory, Cambridge :
1936 *December* 30.

ON THE DISTRIBUTION OF EXTRA-GALACTIC NEBULÆ.

G. C. McVittie, M.A., Ph.D.

Introduction

In some recent papers * Hubble and Tolman have discussed the results of counts of spiral nebulæ and compared them with the theoretical predictions of the expanding universes of general relativity. They arrive at the startling conclusions that space has a positive radius of curvature of the order of $4{\cdot}7 \times 10^8$ light years and that the density of matter is of the order of 10^{-26} gr./cm.^{3}. It is the object of the present note to show that this result is essentially altered by a different method of extrapolating from the observations.

Empirical N, δ Relation

Denoting the number of nebulæ to limiting photographic magnitude m by N, the empirical formula † given by Hubble is

$$\log_{10} N = 0{\cdot}501m - 7{\cdot}371. \qquad (1)$$

This is said to be "probably correct within the uncertainties of the data." In this formula the values of m extend from 18·47 to 21·03. Again the velocity-distance formula is usually given as

$$\log_{10} v = 0{\cdot}2m + 0{\cdot}507 \pm 0{\cdot}012. \qquad (2)$$

If we denote the actual Doppler shift by δ where

$$\delta = \frac{v}{c},$$

we can write, for (2), the equation

$$\log_{10} \delta = 0{\cdot}2m - 4{\cdot}967. \qquad (3)$$

This formula is arrived at by considering magnitudes down to about 17 only. Thus before formulæ (1) and (3) can be used together, some kind of

* E. Hubble and R. C. Tolman, *Astrophys. J.*, 82, 302–337, 1935. E. Hubble, *Astrophys. J.*, 84, 517–554, 1936 December.

† *Astrophys. J.*, 84, 259, 1936.

extrapolation of the latter must be resorted to. Two courses have been adopted :

(*a*) The formula (1) may be accepted as it stands, whilst (3) is either altered in some way or also accepted as it stands. The object of the resulting calculations is then to determine the nature and amount of the curvature of space. The first method is adopted by Hubble and we adopt the latter.

(*b*) Following Sir A. S. Eddington,* we may regard the question of the curvature of space as already settled in favour of a very large *positive* curvature so that all space surveyed in the nebular counts is effectively flat. It is then suggested that the distribution law (1) can be reconciled with this assumption to within the probable errors of the observations.

The observational results are admittedly liable to such large errors that it seems worth while to point out the consequences of assuming that equation (3) *holds without alteration down to magnitude 21.* There is certainly no observational evidence in favour of this extrapolation, but there is equally no evidence against it.

Eliminating, therefore, m from (1) and (3) we obtain

$$\log_{10} N = 2{\cdot}5 \log_{10} \delta + 5{\cdot}046. \tag{4}$$

Hence

$$\frac{1}{N}\frac{dN}{d\delta} = \frac{2{\cdot}5}{\delta}.$$

Taking logarithms again and using (4) we have

$$\log_{10} \frac{dN}{d\delta} = \log_{10} 2{\cdot}5 - \log_{10} \delta + 2{\cdot}5 \log_{10} \delta + 5{\cdot}046,$$

and finally

$$\frac{dN}{d\delta} = 2{\cdot}78 \times 10^5 \delta^{\frac{3}{2}}, \tag{5}$$

which is the equation required. The important feature is the index $\frac{3}{2}$.

Theoretical Formulæ

Taking the metric in the form

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2}\frac{(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2)}{(1 + kr^2/4)^2}, \tag{6}$$

the density and pressure are given by the well-known equations

$$\kappa\rho = -\Lambda + 3\left(\frac{\dot{R}}{R}\right)^2 + \frac{3kc^2}{R^2}, \tag{7}$$

$$\kappa p = \Lambda - \left(\frac{\dot{R}}{R}\right)^2 - \frac{2\ddot{R}}{R} - \frac{c^2 k}{R^2}, \tag{8}$$

* *M.N.*, **97**, 156, 1937.

where dots denote differentiation with respect to t, Λ is the cosmical constant and $\kappa = 8\pi G = 8\pi(6{\cdot}66)10^{-8} = 1{\cdot}674 \times 10^{-6}$. From these we deduce

$$d(\rho R^3) + p dR^3 = 0. \tag{8'}$$

The Doppler effect is given by

$$\delta = \frac{R_0}{R_1} - 1, \tag{9}$$

where R_0 is the value of R at the moment, t_0, of *observation* of a light signal which was *emitted* when R had the value $R_1 = R(t_1)$. Again the number of nebulæ in the shell of radii r and $r + dr$ is *

$$dN = \frac{4\pi\alpha r^2 dr}{(1 + kr^2/4)^3}, \tag{10}$$

where α is the number of nebulæ per unit volume of the subspace with metric

$$du^2 = \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2}{(1 + kr^2/4)^2}.$$

This number is therefore a constant since the nebulæ have fixed co-ordinates (r, θ, ϕ) in this subspace. Lastly, the value of r is given in terms of t_0 and t_1 by the equation of the null geodesic along which the light travels, viz.:

$$\int_0^r \frac{dr}{(1 + kr^2/4)} = c\int_{t_1}^{t_0} \frac{dt}{R(t)}. \tag{11}$$

From (9), (10), (11) W. H. McCrea and K. K. Mitra have calculated the formula †

$$(1+\delta)\frac{dN}{d\delta} = \frac{4\pi\alpha c^3}{\dot{R}_0^3}\left\{\delta^2 + \frac{2}{\dot{R}_0^2}(\ddot{R}_0 R_0 - \dot{R}_0^2)\delta^3 + \frac{1}{12}\frac{1}{\dot{R}_0^4}\begin{pmatrix} 11\dot{R}_0^4 - 46\dot{R}_0^2 R_0 \ddot{R}_0 + 45 R_0^2 \ddot{R}_0^2 \\ - 10\dot{R}_0 \dddot{R}_0 R_0^2 - 4c^2 k \dot{R}_0^2 \end{pmatrix}\delta^4 + \ldots\right\} \tag{12}$$

This formula takes account of the effect of red-shifts on N and δ automatically. It is therefore the most suitable one to compare with the empirical relation (5).

It is also a known result that, if (6) is the metric of the universe, then the rate of recession gives

$$\frac{\dot{R}_0}{R_0} = 2 \times 10^{-17} \text{ sec.}^{-1}. \tag{13}$$

Comparison of Observation and Theory

It is at once clear on comparing (5) and (12) that the empirical relation corresponds to a more rapid increase (since $\delta < 1$) of N with δ than does the first approximation

$$\frac{dN}{d\delta} \sim \delta^2$$

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* W. H. McCrea, *Zeits. f. Astrophysik*, **9**, 302, 1935.

† W. H. McCrea, *loc. cit.*, 303, formula (26).

of (12). We have therefore to inquire to what extent the second and higher approximations of (12) can be made to represent (5).

In the first place the range of the surveys from which (1) is derived corresponds to the five values of m :

$$18{\cdot}47, \quad 19, \quad 19{\cdot}4, \quad 20, \quad 21{\cdot}03.$$

The corresponding values of δ are by (3), approximately,

$$\cdot 05, \quad \cdot 07, \quad \cdot 08, \quad \cdot 11, \quad \cdot 17.$$

Within this range of values of δ the function $\delta^{3/2}$ can be represented by the series

$$\delta^{3/2} = \{6{\cdot}2529\delta^2 - 40{\cdot}059\delta^3 + 75{\cdot}542\delta^4 + 177{\cdot}12\delta^5 - 71{\cdot}275\delta^6\},$$

the accuracy being roughly 3 per cent. Thus (5) is now

$$\frac{dN}{d\delta} = 2{\cdot}78 \times 10^5\{6{\cdot}2529\delta^2 - 40{\cdot}059\delta^3 + 75{\cdot}542\delta^4 + 177{\cdot}12\delta^5 - 71{\cdot}275\delta^6\}. \quad (14)$$

It is only the first *two* terms of (14) which really concern us. For on examining (12) we see that every term involves a new derivative of R, so that, for instance, the coefficient of δ^4 is actually of no use for finding k. We have therefore from (12) and (14)

$$\frac{4\pi\alpha c^3}{\dot{R}_0^{\,3}} = 2{\cdot}78 \times 10^5 \times 6{\cdot}25, \quad (15)$$

$$\frac{2}{\dot{R}_0^{\,2}}(\ddot{R}_0 R_0 - \tfrac{3}{2}\dot{R}_0^{\,2}) = -\frac{40{\cdot}059}{6{\cdot}2529} = -6{\cdot}407. \quad (16)$$

Again, the usual approximation $p = 0$ in (8), yields

$$\Lambda = \frac{\dot{R}_0^{\,2}}{R_0^{\,2}} + \frac{2\ddot{R}_0}{R_0} + \frac{c^2 k}{R_0^{\,2}}. \quad (17)$$

Under these circumstances we may also [equation (8′)] take for the (present) density ρ, the value

$$\rho = \frac{\alpha}{R_0^{\,3}} M, \quad (18)$$

where M is the mass of an average nebula. According to Sinclair Smith the (approximate) value of M * is

$$M = 2 \times 10^{11} \odot = 2 \times 10^{44} \text{ gr.} \quad (19)$$

The equations (13), (15), (18), (19) now yield

$$\rho = \frac{2{\cdot}78 \times 6{\cdot}25 \times 10^5}{4\pi c^3} \left(\frac{\dot{R}_0}{R_0}\right)^3 M$$

$$= 8{\cdot}2 \times 10^{-33} \text{ gr./cm.}^3. \quad (20)$$

* S. Smith, *Astrophys. J.*, **83**, 23, 1936.

Again (13) and (16) yield

$$\frac{\ddot{R}_0}{R_0} = -7 \times 10^{-34} \text{ sec.}^{-2}. \tag{21}$$

The expansion is therefore being *retarded.*

Lastly, to find k, R_0 and Λ we proceed as follows :—
Eliminate Λ from (17) and (7), so that

$$\frac{2kc^2}{R_0^2} = \kappa\rho - 2\left(\frac{\dot{R}_0}{R_0}\right)^2 + 2\frac{\ddot{R}_0}{R_0},$$

whence by (20), (21), (13) and the value of $\kappa(= 1{\cdot}674 \times 10^{-6})$, we have

$$\frac{2kc^2}{R_0^2} = -22 \times 10^{-34} + 1{\cdot}674 \times 8{\cdot}2 \times 10^{-39}.$$

We may clearly neglect the second term compared with the first. Hence

$$k = -1, \tag{22}$$

$$R_0 = 9 \times 10^{26} \text{ cm.} = 9{\cdot}4 \times 10^8 \text{ lt. yrs.} \tag{23}$$

We conclude that space is *hyperbolic* and therefore *infinite in extent.* But its radius nevertheless has the comparatively small value of some 10^9 light years.

The cosmical constant is obtained from (17), using (13), (21), (22) and (23). Its value is

$$\Lambda = -21 \times 10^{-34} \text{ sec.}^{-2}, \tag{24}$$

and is thus negative.

Conclusion

The nebular counts are therefore in agreement with an infinite universe of *negative* curvature whose radius is of the order of 10^9 light years. The average density of matter is of the order of 10^{-32} gr./cm.^{3}, the cosmical constant has the negative value -21×10^{-34} sec.$^{-2}$ and the expansion is being retarded.

Hubble's observations, however, suggest a *maximum* value of ·77 (instead of the value ·507 in (2)) for the constant term in the relation between velocity and apparent magnitude *uncorrected for red-shift*, which is alone appropriate for a comparison of theory and observation. With this value of the constant the present method yields

$$\frac{\ddot{R}_0}{R_0} = -1{\cdot}4 \times 10^{-34} \text{ sec.}^{-2}; \quad \rho = 2{\cdot}2 \times 10^{-33} \text{ gr./cm.}^3; \quad k = -1;$$

$$R_0 = 14 \times 10^8 \text{ lt. yrs.}; \quad \Lambda = -4 \times 10^{-34} \text{ sec.}^{-2};$$

in good agreement with the previous results.

King's College,
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1937 *January* 13.

THE ORBIT OF Σ 2026.

R. v. d. R. Woolley and L. S. T. Symms.

(Communicated by the Astronomer Royal)

This binary star (A.D.S. 9982) has been observed frequently since its discovery in 1830 by Struve, when the two components (magnitudes 8·6 and 9·1) were separated by 2″·5. Various observers measured the separation and position-angle, but the observations before 1890 are unsatisfactory in both co-ordinates, particularly in separation. After 1890 the star was measured by good observers under more or less modern conditions, but the separation had decreased to about half a second of arc, and there is still an unsatisfactory scatter in the measured distances. Periastron passage occurred at 1906·8, and shortly after this date we find an extremely satisfactory set of measures extending up to the present day and presenting a high degree of self-consistency. An orbit was published by Comstock in 1918,* but this orbit is noted by Aitken (A.D.S.) as giving errors in position-angle of about 6 degrees. It is clear from what has been said above that the portion of the orbit which has been reliably observed has increased substantially since Comstock's orbit was computed, and that an orbit can be computed to-day which will not be bettered for many decades, since the period is very long.

We propose to set out the computation of an orbit with more detail than sometimes appears, although the method is not original, since it follows the work of Innes, van den Bos and Finsen ; however, a permanent record of the treatment may be of some use to computers.

To begin with, an attempt was made to compute the orbit by Russell's method.† Despite a large number of trials, in which various modifications were made in the adopted positions, no preliminary curves could be found from which the elements of the apparent ellipse could be deduced with confidence, and in every case when an ephemeris was computed it was found impossible to represent the observed position-angles round periastron. Our first attempts gave discordances as large as 15° in position-angle, but the orbit was improved by making arbitrary alterations in the elements until these discordances (observed minus computed position-angle) ranged from about $+3°\cdot5$ to $-3°\cdot5$. It was decided at this stage to apply differential equations to improve the elements of the orbit. Ten places were adopted as representing the observations as well as possible. In adopting distances before periastron it was necessary to pay far more attention to the position-angles and law of areas than to the observations of distance : in other words, position-angle seems to have been more reliably observed than distance.

* *A.J.*, **31**, 33, 1918. Comstock gives four orbits besides his own with periods ranging from 522 years to 140 years.

† *M.N.*, **93**, 599, 1933.

Differential corrections to the elements of an orbit are very conveniently applied by following the methods of Innes, van den Bos and Finsen.* The rectangular co-ordinates of any point on the apparent ellipse can be expressed in the form

$$x = AX + FY,$$
$$y = BX + GY,$$

where (A, B) are the rectangular co-ordinates of the periastron point and (F, G) are the co-ordinates of the projection on the plane of the apparent ellipse of that point on the auxiliary circle to the true orbit for which the eccentric anomaly is 90° ; in other words, the point (F, G) can be found by producing the diameter conjugate to periastron to cut the projection of the auxiliary circle.

The quantities X and Y are simple functions of the eccentric anomaly E and eccentricity e, being given by

$$X = \cos E - e,$$
$$Y = \sqrt{1 - e^2} \sin E.$$

These quantities are tabulated in *Union Observatory Circular*, 71 (1926), in terms of the mean anomaly M, so that a solution of Kepler's equation is embodied in the tables. Entries are made for every degree of M from 0° to 180° and for values of e from 0·00 to 1·00 in steps of 0·01.

In the present case the preliminary orbit was found by Russell's method, which may well be used for this purpose. This method gives the apparent ellipse in the form

$$y = a + bx \pm c\sqrt{(x_1 - x)(x - x_0)},$$

from which the centre (ξ, η) can be computed. The straight line passing through the centre (ξ, η) and the origin passes through the periastron point (x', y'), which is best found graphically. The Innes co-ordinates are then found from the four equations

$$A = x' - \xi,$$
$$B = y' - \eta,$$
$$F = (bA - B) \sec \phi / c,$$
$$G = bF + cA \sec \phi,$$

where $\sin \phi = e$, the eccentricity.

Differential corrections to the Innes elements are given by

$$\Delta x = X\Delta A + Y\Delta F + P_x \Delta e + Q_x(n\Delta T - (t - T)\Delta n),$$
$$\Delta y = X\Delta B + Y\Delta G + P_y \Delta e + Q_y(n\Delta T - (t - T)\Delta n),$$

where

$$P_x = A\frac{\partial X}{\partial e} + F\frac{\partial Y}{\partial e}, \qquad Q_x = -A\frac{\partial X}{\partial M} - F\frac{\partial Y}{\partial M};$$

$$P_y = B\frac{\partial X}{\partial e} + G\frac{\partial Y}{\partial e}, \qquad Q_y = -B\frac{\partial X}{\partial M} - G\frac{\partial Y}{\partial M}.$$

* *Union Obs. Circ.*, 68, 1926.

The partial derivatives $\frac{\partial X}{\partial e}$, $\frac{\partial X}{\partial M}$, etc., are very conveniently found in the tables.

The Innes elements A, B, F, G, e, T, n are in themselves a complete specification of the apparent ellipse, and as van den Bos emphasises, such elements as Ω in the usual representation are without physical significance. In some researches, however, it may be convenient to have the semiaxis major a and the inclination i already computed. They can be found from the Innes elements by the equations

$$a^2(1+\cos^2 i) = A^2+B^2+F^2+G^2$$
$$a^2 \cos i = AG - BF.$$

To return to Σ 2026, it was found after repeated attempts to obtain a satisfactory result by Russell's graphical method that the following orbit I gave a fairly close fit, and it was adopted for the first approximation:

$A = -3{\cdot}100,$ $\quad e = 0{\cdot}86,$
$B = -0{\cdot}235,$ $\quad T = 1907{\cdot}1,$
$F = +0{\cdot}374,$ $\quad n = 0{\cdot}00917$ radians/year.
$G = +2{\cdot}060,$

Normal places were adopted at ten points (from 1853 to 1932) and twenty equations for the observed minus computed co-ordinates Δx and Δy were obtained. A least square solution was then made, which gave the results (n being counted in degrees per year):

$$\Delta A = +{\cdot}123 \pm 0{\cdot}38,$$
$$\Delta B = -{\cdot}042 \pm 0{\cdot}036,$$
$$\Delta F = -{\cdot}136 \pm 0{\cdot}062,$$
$$\Delta G = -{\cdot}011 \pm 0{\cdot}054,$$
$$\Delta e = -{\cdot}0017 \pm 0{\cdot}0028,$$
$$n\Delta T = -{\cdot}141 \pm 0{\cdot}076,$$
$$\Delta n = +{\cdot}0033 \pm 0{\cdot}0039.$$

These corrections were applied to the elements of orbit I to give the revised orbit II.

It will be seen that low weight is attached to the corrections to ΔA and Δn, because the observations do not define the apastron end of the ellipse. There is no substantial change in e. In this case no second approximation is necessary.

The following table shows the comparison between the adopted places and the orbits I and II (the revised orbit):—

Date	Adopted Place		$\theta_a - \theta_c$		$\rho_a - \rho_c$	
	ρ_a	θ_a	I	II	I	II
	″	°	°	°	″	″
1853·8	1·90	333·7	+0·4	−0·1	−0·3	0·0
1870·9	1·36	322·0	−0·1	−1·0	−0·1	0·0
1880·5	1·04	311·0	+0·2	−0·8	0·0	+0·1
1890·1	0·71	290·9	+1·9	+0·2	0·0	0·0
1899·5	0·50	245·0	+3·5	+2·1	0·0	+0·2

Date	Adopted Place		$\theta_a - \theta_c$		$\rho_a - \rho_c$	
	ρ_a	θ_a	I	II	I	II
	″	°	°	°	″	″
1907·1	0·43	181·1	−3·3	−1·7	0·0	+0·2
1912·8	0·40	120·6	−0·7	+0·7	+0·2	+0·2
1920·4	0·66	70·6	+3·5	+1·5	−0·2	−0·1
1926·1	0·92	55·0	+2·5	−0·2	−0·5	−0·4
1931·8	1·22	47·1	+2·5	−0·1	−0·3	0·0

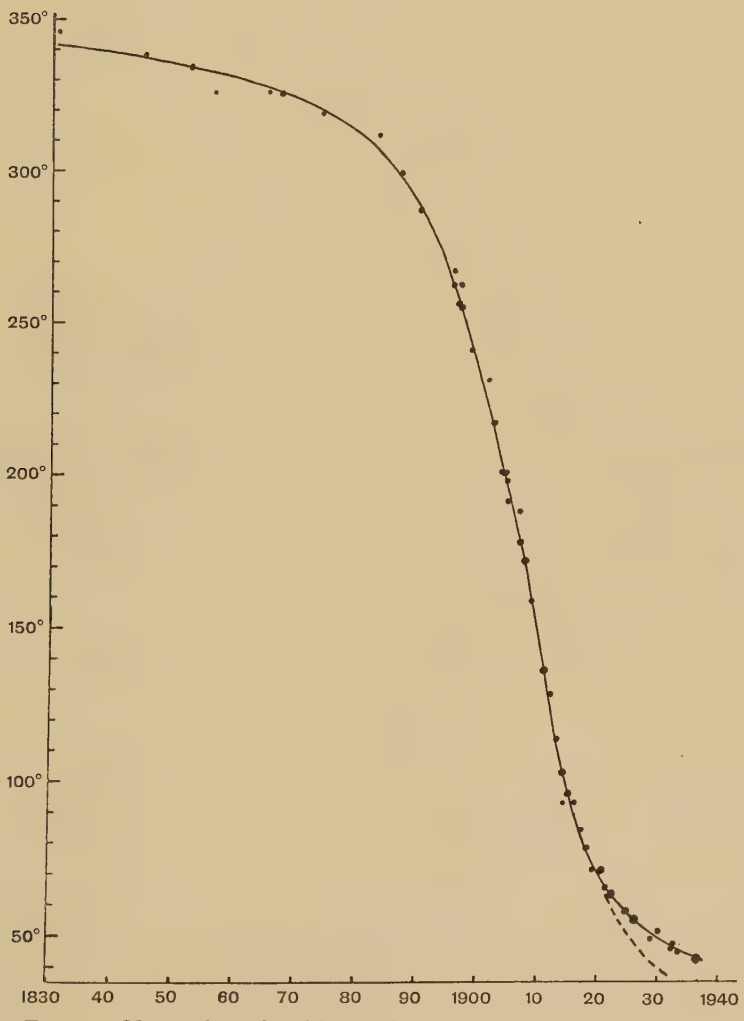

FIG. 1.—*Observations of position-angle compared with present orbit II.*

The values of $\rho_a - \rho_c$ from 1880·5 to 1907·1 were left zero in adopting normal places because the observations were inadequate. It does not seem worth while attempting to improve upon the present "revised" orbit II. The accompanying figures show the actual observations of position-angle and separation plotted against time and compared with the "revised" orbit

and also with part of Comstock's orbit (shown by a dotted line). It will be seen that the agreement between the present orbit II and the observations to date is as close as could be desired.

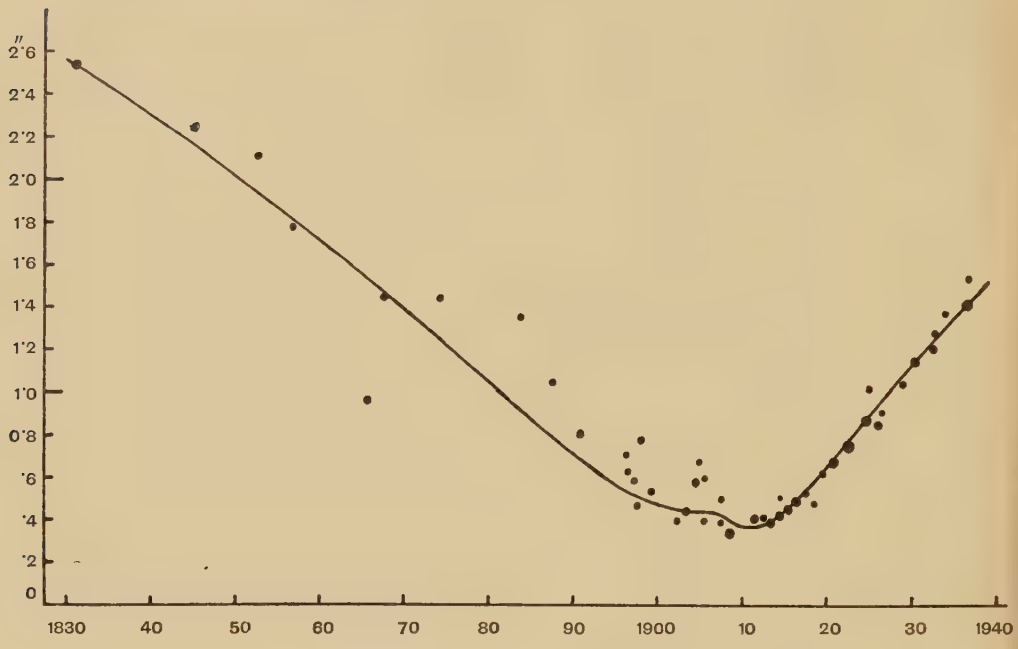

FIG. 2.—*Observations of separation compared with present orbit II.*

The concluded elements (orbit II) are

$$A = -2{\cdot}977,$$
$$B = -0{\cdot}277,$$
$$F = +0{\cdot}238,$$
$$G = +2{\cdot}049,$$
$$e = \quad 0{\cdot}86,$$
$$T = \quad 1906{\cdot}8,$$
$$n = \quad 0{\cdot}00923.$$

We may add for convenience:

Period = 680 years,
Semiaxis major = 3″·04,
Inclination = ±130°·6.

The dynamical parallax from these elements is 0″·032.

Ephemeris

	°	″
1930·0	49·4	1·14
35·0	44·1	1·36
40·0	40·1	1·58
45·0	37·3	1·76
50·0	34·9	1·95
55·0	32·9	2·12
1960·0	31·2	2·29

We may conclude this note by remarking that a great deal of the work which needs to be done on double star orbits in the Northern hemisphere is of the nature of improving an existing orbit in the light of later observations, and that the differential method of Innes and van den Bos is admirably suited to this work. Naturally the differential method is at its best when the original orbital elements have been taken fairly close to the ultimate ones, and where the published orbit has shown itself to be sadly astray it will be desirable to compute a preliminary orbit, perhaps by Russell's method.

A METHOD OF DERIVING THE GENERAL VELOCITY ELLIPSOID FROM PROPER MOTIONS.

J. Fleming, B.A.

(Communicated by W. M. Smart)

1. *Statement of the Problem.*—On the assumption of a spheroidal theory of the distribution of stellar velocities Schwarzschild has given two methods of determining, from proper motion data alone, the constants involved and also the position of the solar apex. The present problem is to discover a method of finding the constants and the position of the solar apex on the basis of a general ellipsoidal distribution. It will be noticed that this more general assumption necessitates the determination of two extra constants.

2. *Hypothesis.*—Let OX, OY, OZ be the usual set of equatorial axes; that is to say, OX points to the vernal equinox and OZ to the North Pole. Let U, V, W be the components of peculiar motion of a star relative to these axes. Let dN be the number of stars having component velocities between U, $U+dU$; V, $V+dV$; W, $W+dW$. We shall suppose that

$$dN = Ke^{-F(U,\,V,\,W)}dUdVdW, \tag{2.1}$$

where

$$F(U, V, W) = AU^2 + BV^2 + CW^2 + 2FVW + 2GWU + 2HUV. \tag{2.2}$$

If N is the total number of stars considered it follows from integration and invariancy that

$$K = \frac{N\sqrt{\Delta}}{\pi^{3/2}},$$

where

$$\Delta = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}.$$

3. *Transformation of Axes.*—Consider a small region of the sky near the point $S(\alpha, \delta)$ on the celestial sphere. Let the plane OZS cut the plane OXY along ON and let OM be drawn in the plane OXY perpendicular to ON. Then

$$\angle XON = \alpha, \qquad \angle NOS = \delta.$$

We may consider a star at S as having velocity components u, v, w, where u is directed along the great-circle towards Z, v along that towards M and w is radial. Then

$$\left.\begin{aligned} U &= l_1u + m_1v + n_1w, \\ V &= l_2u + m_2v + n_2w, \\ W &= l_3u + m_3v + n_3w, \end{aligned}\right\} \tag{3.1}$$

where

$$\left.\begin{aligned} l_1 &= -\cos\alpha\sin\delta, & m_1 &= -\sin\alpha, & n_1 &= \cos\alpha\cos\delta, \\ l_2 &= -\sin\alpha\sin\delta, & m_2 &= +\cos\alpha, & n_2 &= \sin\alpha\cos\delta, \\ l_3 &= \cos\delta, & m_3 &= 0, & n_3 &= \sin\delta. \end{aligned}\right\} \tag{3.2}$$

Hence

$$F(U, V, W) \equiv AU^2 + BV^2 + CW^2 + 2FVW + 2GWU + 2HUV$$

becomes, after transformation,

$$F(U, V, W) = au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv. \tag{3.3}$$

If Δ, δ, P represent the following matrices,

$$\Delta = \begin{pmatrix} A & H & G \\ H & B & F \\ G & F & C \end{pmatrix}, \qquad \delta = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, \qquad P = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix},$$

then by a well-known theorem

$$\delta = P\Delta P^{-1}. \tag{3.4}$$

4. *Algebraic Properties of Transformations.*—We shall require expressions for the minors of a, b, c, f, g, h in $||\ \delta\ ||$, the determinant corresponding to δ, in terms of the minors of A, B, C, F, G, H in $||\ \Delta\ ||$.

By the theory of matrices if $\bar{a}$ is the minor of a, $\bar{A}$ that of A, and so on and

$$\bar{\Delta} = \begin{pmatrix} \bar{A} & \bar{H} & \bar{G} \\ \bar{H} & \bar{B} & \bar{F} \\ \bar{G} & \bar{F} & \bar{C} \end{pmatrix}, \qquad \bar{\delta} = \begin{pmatrix} \bar{a} & \bar{h} & \bar{g} \\ \bar{h} & \bar{b} & \bar{f} \\ \bar{g} & \bar{f} & \bar{c} \end{pmatrix},$$

then

$$\Delta\bar{\Delta} = ||\ \Delta\ || = k, \text{ say,}$$

and

$$\begin{aligned} \delta\bar{\delta} = ||\ \delta\ || &= ||\ P\Delta P^{-1}\ || \\ &= ||\ \Delta\ || \\ &= k. \end{aligned}$$

Hence

$$\bar{\Delta} = k\Delta^{-1}, \qquad \bar{\delta} = k\delta^{-1}. \tag{4.1}$$

But from equation (3.4)

$$\begin{aligned} \delta^{-1} &= (P\Delta P^{-1})^{-1} \\ &= P\Delta^{-1}P^{-1} \end{aligned}$$

and so

$$\bar{\delta} = P\bar{\Delta}P^{-1}. \tag{4.2}$$

Equating corresponding terms in the matrix equation (4.2) we obtain

$$\left.\begin{aligned}\bar{a} &= l_1^2\bar{A} + l_2^2\bar{B} + l_3^2\bar{C} + 2l_2l_3\bar{F} + 2l_3l_1\bar{G} + 2l_1l_2\bar{H},\\ \bar{b} &= m_1^2\bar{A} + m_2^2\bar{B} + m_3^2\bar{C} + 2m_2m_3\bar{F} + 2m_3m_1\bar{G} + 2m_1m_2\bar{H},\\ \bar{h} &= l_1m_1\bar{A} + l_2m_2\bar{B} + l_3m_3\bar{C}\\ &\quad + (l_2m_3 + l_3m_2)\bar{F} + (l_3m_1 + l_1m_3)\bar{G} + (l_1m_2 + l_2m_1)\bar{H}.\end{aligned}\right\} \tag{4.3}$$

These are the expressions which will be required later.

Equations (4.3) may also be obtained by straightforward algebraic methods.

5. *Peculiar Transverse Motions.*—Let us consider n stars in a region S. Suppose that there are dn stars which have transverse components between u, $u+du$; v, $v+dv$. Now

$$dUdVdW = dudvdw.$$

Hence

$$dn = \frac{n\sqrt{\Delta}}{\pi^{3/2}} e^{-au^2-bv^2-2huv} dudv \int_{-\infty}^{\infty} e^{-cw^2-2w(fv+gu)} dw.$$

But

$$\int_{-\infty}^{\infty} e^{-cw^2-2w(fv+gu)} dw = \sqrt{\frac{\pi}{c}}\, e^{\frac{(fv+gu)^2}{c}}.$$

Hence

$$dn = \frac{n}{\pi}\sqrt{\frac{\Delta}{c}} dudv\, e^{-au^2-bv^2-2huv+\frac{1}{c}(fv+gu)^2}.$$

This result may be written in the more convenient form

$$dn = \frac{n}{\pi}\sqrt{\frac{\Delta}{c}} dudv\, e^{-pu^2-qv^2-2suv}, \tag{5.1}$$

where

$$\left.\begin{aligned} p &= \frac{1}{c}(ac - g^2) = \frac{1}{c}b,\\ q &= \frac{1}{c}(bc - f^2) = \frac{1}{c}\bar{a},\\ s &= \frac{1}{c}(ch - fg) = -\frac{1}{c}\bar{h}.\end{aligned}\right\} \tag{5.2}$$

6. *Observed Transverse Motions.*—Suppose that $(-U_0, -V_0)$ represents the projection of the solar motion on the tangent plane at S, then each star will have peculiar components u, v and parallactic components U_0, V_0.

If x, y are the components of the observed linear motion, then

$$\left.\begin{aligned} x &= u + U_0,\\ y &= v + V_0.\end{aligned}\right\} \tag{6.1}$$

Substituting equations (6.1) in equation (5.1) we have

$$dn = \frac{n}{\pi}\sqrt{\frac{\Delta}{c}} dxdy\, e^{-p(x-U_0)^2-q(y-V_0)^2-2s(x-U_0)(y-V_0)}. \tag{6.2}$$

We now put

$$\begin{aligned} x &= r\cos\phi,\\ y &= r\sin\phi.\end{aligned}$$

It follows then from equation (6.2) that the number of stars moving in the sector ϕ to $\phi+d\phi$ having total transverse motions between r and $r+dr$ is

$$\frac{n}{\pi}\sqrt{\frac{\Delta}{c}}r dr d\phi\, e^{-p(r\cos\phi-U_0)^2-q(r\sin\phi-V_0)^2-2s(r\cos\phi-U_0)(r\sin\phi-V_0)}.$$

Now let $n(\phi)d\phi$ be the total number of stars moving in the sector ϕ to $\phi+d\phi$, then by integrating with respect to r from 0 to ∞ we have

$$n(\phi)=\frac{n}{\pi}\sqrt{\frac{\Delta}{c}}e^{-pU_0^2-qV_0^2-2sU_0V_0}\int_0^\infty r dr\, e^{-\rho^2r^2-2\rho\xi r},$$

where

$$\rho^2=p\cos^2\phi+q\sin^2\phi+2s\cos\phi\sin\phi, \tag{6.3}$$

$$\xi=\{(pU_0+sV_0)\cos\phi+(sU_0+qV_0)\sin\phi\}/\rho. \tag{6.4}$$

We now put

$$\tau=\rho r-\xi,$$

then

$$n(\phi)=\frac{n}{\pi}\sqrt{\frac{\Delta}{c}}e^{-pU_0^2-qV_0^2-2sU_0V_0}\frac{e^{\xi^2}}{\rho^2}\int_{-\xi}^\infty(\tau+\xi)e^{-\tau^2}d\tau$$

$$=\frac{\mathcal{J}^2}{\rho^2}f(\xi), \tag{6.5}$$

where

$$f(\xi)=\frac{2}{\sqrt{\pi}}\left(\tfrac{1}{2}+\xi e^{\xi^2}\int_{-\xi}^\infty e^{-\tau^2}d\tau\right), \tag{6.6}$$

and

$$\mathcal{J}^2=\frac{n}{2}\sqrt{\frac{\Delta}{\pi c}}e^{-pU_0^2-qV_0^2-2sU_0V_0}. \tag{6.7}$$

7. *Transformation to a New Angular Parameter.*—By equation (6.5) we have that the number of stars moving in directions between ϕ and $\phi+d\phi$ is

$$n(\phi)d\phi=\frac{\mathcal{J}^2}{\rho^2}f(\xi)d\phi.$$

Let $n'(\phi)d\phi$ be the number of stars moving in directions between $\phi+\pi$ and $\phi+\pi+d\phi$. By inspection of equations (6.3) and (6.4) it is seen that

$$n'(\phi)d\phi=\frac{\mathcal{J}^2}{\rho^2}f(-\xi)d\phi.$$

Now

$$f(\xi)-f(-\xi)=\frac{2}{\sqrt{\pi}}\left\{\tfrac{1}{2}+\xi e^{\xi^2}\int_{-\xi}^\infty e^{-\tau^2}d\tau\right\}-\frac{2}{\sqrt{\pi}}\left\{\tfrac{1}{2}-\xi e^{\xi^2}\int_{\xi}^\infty e^{-\tau^2}d\tau\right\}$$

$$=2\xi e^{\xi^2}.$$

Hence

$$\{n(\phi)-n'(\phi)\}d\phi=\frac{\mathcal{J}^2}{\rho^2}d\phi\,.\,2\xi e^{\xi^2}. \tag{7.1}$$

Now by equation (6.4)

$$\xi = \frac{(pU_0 + sV_0)\cos\phi + (sU_0 + qV_0)\sin\phi}{\rho}$$

and so

$$d\xi = \frac{d\phi}{\rho^2}\Big[\rho\{-(pU_0+sV_0)\sin\phi + (sU_0+qV_0)\cos\phi\} - \frac{d\rho}{d\phi}\{(pU_0+sV_0)\cos\phi + (sU_0+qV_0)\sin\phi\}\Big]$$

$$= \frac{d\phi}{\rho^2}\cdot\frac{1}{\rho}[\{p\cos^2\phi + q\sin^2\phi + 2s\cos\phi\sin\phi\}\{-(pU_0+sV_0)\sin\phi + (sU_0+qV_0)\cos\phi\} - \{(q-p)\cos\phi\sin\phi + s(\cos^2\phi - \sin^2\phi)\}\{(pU_0+sV_0)\cos\phi + (sU_0+qV_0)\sin\phi\}]$$

$$= \frac{d\phi}{\rho^2}\cdot\frac{1}{\rho}(pq - s^2)(V_0\cos\phi - U_0\sin\phi). \tag{7.2}$$

Now put

$$\eta = \frac{\sqrt{pq - s^2}(V_0\cos\phi - U_0\sin\phi)}{\rho}, \tag{7.3}$$

then

$$\eta^2 + \xi^2 = \frac{1}{\rho^2}[(pq - s^2)(V_0^2\cos^2\phi - 2U_0V_0\cos\phi\sin\phi + U_0^2\sin^2\phi) + (pU_0+sV_0)^2\cos^2\phi + 2(pU_0+sV_0)(sU_0+qV_0)\cos\phi\sin\phi + (sU_0+qV_0)^2\sin^2\phi]$$

$$= \frac{1}{\rho^2}[U_0^2(pq\sin^2\phi - s^2\sin^2\phi + p^2\cos^2\phi + 2ps\cos\phi\sin\phi + s^2\sin^2\phi) + 2U_0V_0(s^2\cos\phi\sin\phi - pq\cos\phi\sin\phi + ps\cos^2\phi + pq\cos\phi\sin\phi + s^2\cos\phi\sin\phi + qs\sin^2\phi) + V_0^2(pq\cos^2\phi - s^2\cos^2\phi + s^2\cos^2\phi + 2qs\cos\phi\sin\phi + q^2\sin^2\phi)]$$

$$= pU_0^2 + 2sU_0V_0 + qV_0^2,$$

by equation (6.3),

$$= \mu^2, \text{ say.} \tag{7.4}$$

Substituting equations (7.2) and (7.3) in equation (7.1) we have that

$$\{n(\phi) - n'(\phi)\}d\phi = 2\mathfrak{F}^2\xi e^{\xi^2}\frac{d\xi}{\eta\sqrt{pq - s^2}}. \tag{7.5}$$

We now make a further transformation. Since

$$\eta^2 + \xi^2 = \mu^2,$$

where μ is independent of ϕ, hence

$$\eta d\eta = -\xi d\xi.$$

Hence equation (7.5) may be written

$$\{n(\phi) - n'(\phi)\}d\phi = -\frac{2\mathfrak{F}^2 e^{\mu^2 - \eta^2}d\eta}{\sqrt{pq - s^2}}. \tag{7.6}$$

Now by equation (6.7)

$$\mathcal{J}^2 = \frac{n}{2}\sqrt{\frac{\Delta}{\pi c}}e^{-pU_0^2-qV_0^2-2sU_0V_0}$$
$$= \frac{n}{2}\sqrt{\frac{\Delta}{\pi c}}e^{-\mu^2}, \qquad \text{by equation (7.4).}$$

Furthermore, by equations (5.2)

$$pq - s^2 = \frac{1}{c^2}(\bar{a}b - \bar{h}^2) = \frac{\Delta}{c}.$$

Hence equation (7.6) may be written

$$\{n(\phi) - n'(\phi)\}d\phi = -\frac{n}{\sqrt{\pi}}e^{-\eta^2}d\eta. \tag{7.7}$$

8. *Counts of Stars moving in Semicircular Sectors.*—We now suppose that $N(\phi)$ is the number of stars moving in directions between ϕ and $\phi+\pi$ and also that $N'(\phi)$ is the number moving in directions between $\phi+\pi$ and $\phi+2\pi$. It follows that

$$N(\phi) - N'(\phi) = \int_{\phi}^{\phi+\pi}\{n(\phi) - n'(\phi)\}d\phi$$
$$= -\frac{n}{\sqrt{\pi}}\int e^{-\eta^2}d\eta, \tag{8.1}$$

by equation (7.7), where the integration extends from $\phi = \phi$ to $\phi = \phi+\pi$.

From equations (7.3) and (7.4) it follows that η is a periodic function of ϕ of period 2π and that it varies between μ and $-\mu$. Furthermore, the value of η at $\phi+\pi$ is equal in magnitude but opposite in sign to its value at ϕ. Hence the limits of integration in equation (8.1) are η and $-\eta$. It follows then that

$$N(\phi) - N'(\phi) = \frac{2n}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}d\eta. \tag{8.2}$$

Introducing the error function

$$\Theta(\eta) = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}d\eta, \tag{8.3}$$

and writing

$$M(\phi) = N(\phi) - N'(\phi), \tag{8.4}$$

we have

$$M(\phi) = n\Theta(\eta). \tag{8.5}$$

9. *Determination of the Solar Motion.*—We now plot $M(\phi)$ against ϕ. From the graph (fig. 1) we find the value of ϕ for which $M(\phi)$ vanishes; let it be ϕ_1. It follows from equation (8.5) that η vanishes, and referring back to equation (7.3) we find that

$$\frac{U_0}{V_0} = \cot\phi_1 = \lambda, \text{ say.} \tag{9.1}$$

Now let $-U_1$, $-V_1$, $-W_1$ be the components of the solar motion referred to the equatorial axes OX, OY, OZ. Then by equations (3.1)

$$\left.\begin{aligned} U_0 &= l_1U_1 + l_2V_1 + l_3W_1, \\ V_0 &= m_1U_1 + m_2V_1 + m_3W_1 \end{aligned}\right\}. \tag{9.2}$$

From equations (9.1) and (9.2) we obtain

$$U_1(l_1 - \lambda m_1) + V_1(l_2 - \lambda m_2) + W_1(l_3 - \lambda m_3) = 0. \tag{9.3}$$

Analysing a number of areas we obtain an equation of the type (9.3) from each. Solving by least squares we can find the apex of the solar motion.

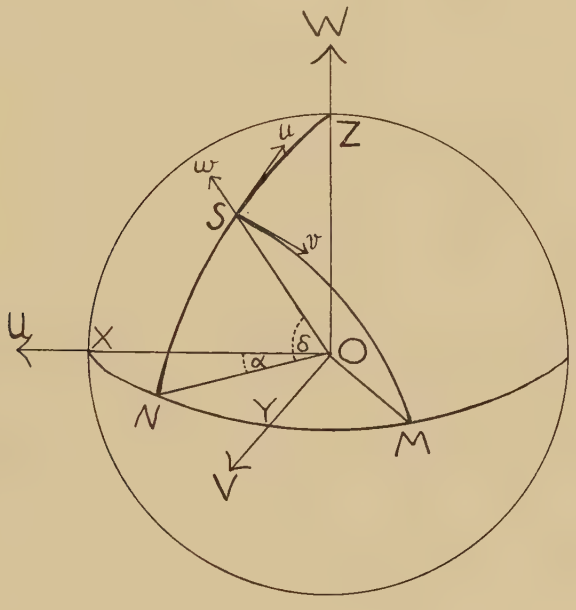

Fig. 1.

If we suppose that the magnitude of the solar motion is known from other sources we can find U_0, V_0 for each region.

10. *Determination of the Ellipsoidal Constants.*—From fig. 1 we can determine the value of ϕ for which $M(\phi)$ is a maximum; let it be ϕ_2. It follows from equation (8.5) that η is then a maximum; that is to say, by equation (7.4) it is equal to μ, and so ξ vanishes. Hence, by equation (6.4),

$$-\frac{sU_0 + qV_0}{pU_0 + sV_0} = \cot\phi_2 = \sigma, \text{ say.} \tag{10.1}$$

By equations (5.2) and (9.1) we have that

$$\sigma\lambda\bar{b} + \bar{a} - (\sigma + \lambda)\bar{h} = 0. \tag{10.2}$$

Using equations (4.3) we obtain

$$\bar{A}\{l_1^2-(\sigma+\lambda)l_1m_1+\sigma\lambda m_1^2\}+\bar{B}\{l_2^2-(\sigma+\lambda)l_2m_2+\sigma\lambda m_2^2\}$$
$$+\bar{C}\{l_3^2-(\sigma+\lambda)l_3m_3+\sigma\lambda m_3^2\}+\bar{F}\{l_2l_3-(\sigma+\lambda)(l_2m_3+l_3m_2)+\sigma\lambda m_2m_3\}$$
$$+\bar{G}\{l_3l_1-(\sigma+\lambda)(l_3m_1+l_1m_3)+\sigma\lambda m_3m_1\}+\bar{H}\{l_1l_2-(\sigma+\lambda)(l_1m_2+l_2m_1)$$
$$+\sigma\lambda m_1m_2\}=0. \quad (10.3)$$

An equation of the type (10.3) can be obtained from each region. If we take a sufficient number of regions we can solve by least squares and obtain $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{F}$, $\bar{G}$, $\bar{H}$ in terms of some theoretical unit.

We can now find A, B, C, F, G, H. For if

$$\bar{\Delta}=\begin{vmatrix}\bar{A}&\bar{H}&\bar{G}\\ \bar{H}&\bar{B}&\bar{F}\\ \bar{G}&\bar{F}&\bar{C}\end{vmatrix} \quad \text{and} \quad \Delta=\begin{vmatrix}A&H&G\\ H&B&F\\ G&F&C\end{vmatrix},$$

then

$$\bar{B}\bar{C}-\bar{F}^2=A\Delta$$

and

$$\begin{aligned}\bar{\Delta}&=\bar{A}(\bar{B}\bar{C}-\bar{F}^2)+\bar{H}(\bar{F}\bar{G}-\bar{C}\bar{H})+\bar{G}(\bar{H}\bar{F}-\bar{B}\bar{G})\\ &=\bar{A}A\Delta+\bar{H}H\Delta+\bar{G}G\Delta\\ &=\Delta^2.\end{aligned}$$

Hence

$$A=\frac{1}{\sqrt{\bar{\Delta}}}(\bar{B}\bar{C}-\bar{F}^2), \quad (10.4)$$

and so on. We notice that the units in which A, B, C, F, G, H are expressed are still theoretical and we have not yet found their connection with practical units.

Now consider some particular region again, we can find a, b, c, f, g, h and therefore p, q, s in terms of our theoretical units. By equation (8.5) we have

$$\text{maximum value of } M(\phi)=n\Theta(\mu).$$

From the graph of $M(\phi)$ we can find μ. But

$$pU_0^2+2sU_0V_0+qV_0^2=\mu^2,$$

where $\frac{1}{\sqrt{p}}$, $\frac{1}{\sqrt{q}}$, $\frac{1}{\sqrt{s}}$, U_0, V_0 are expressed in terms of kilometres per second. We can therefore find the connection between our theoretical units and kilometres per second and so find the absolute values of A, B, C, F, G, H.

We have now calculated the velocity ellipsoid

$$AU^2+BV^2+CW^2+2FVW+2GWU+2HUV=1. \quad (10.5)$$

The lengths and directions of its principal axes can be found by a well-known method. See, for example, D. M. Y. Sommerville, *Analytical Geometry in Three Dimensions*, 1934, p. 166, or R. J. T. Bell, *Co-ordinate Geometry of Three Dimensions*, 1928, p. 204.

11. *Example.*—As an example of the method Group I of the Cambridge Proper Motions * is analysed.

* See W. M. Smart, *M.N.*, **87**, 122, 1927.

The group chosen consists of 545 stars lying in a small region of the sky centred on the point R.A. 4° 7′, Dec. 50°. Table I gives the values of $n(\phi)$, $n'(\phi)$, $N(\phi)$, $N'(\phi)$ and $M(\phi)$ for various values of the position-angle ϕ.

TABLE I

ϕ	$n(\phi)$	$n'(\phi)$	$N(\phi)$	$N'(\phi)$	$M(\phi)$
°					
10	7	29	393	152	241
20	3	23	415	130	285
30	7	19	435	110	325
40	7	14	447	98	349
50	6	16	454	91	363
60	14	12	464	81	383
70	16	4	462	83	379
80	27	5	450	95	355
90	34	4	428	117	311
100	50	3	398	147	251
110	40	3	351	194	157
120	40	3	314	231	83
130	33	3	277	268	9
140	28	3	247	298	− 51
150	31	1	222	323	−101
160	20	3	192	353	−161
170	15	2	175	370	−195
180	15	5	162	383	−221

From the graph (fig. 2) the required results can be read off. In this case it is found that

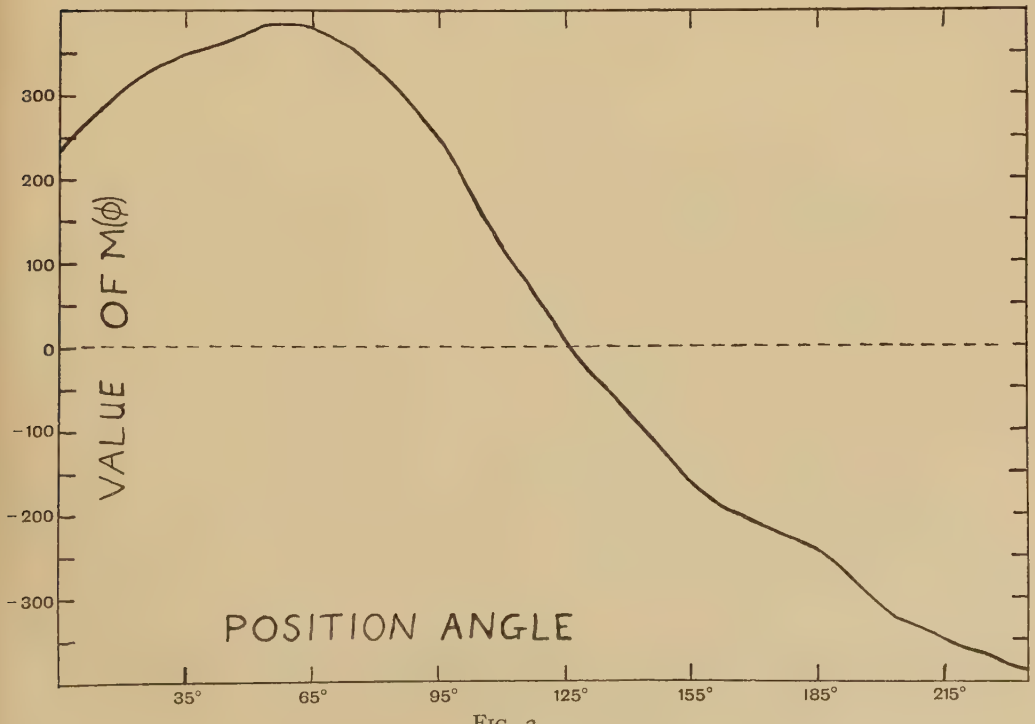

FIG. 2.

$$\phi_1 = 126°,$$
$$\phi_2 = 58°,$$

and maximum value of $M = 384$.

Taking the solar apex found by Smart * for these stars, viz.

R.A. 273°·6, Dec. 43°·6,

the calculated value of the position-angle ϕ_1 is

$$\phi_1 = 121°,$$

showing good agreement with the result found by the present method.

The analysis of Smart's eight regions by the present method gave the following co-ordinates for the solar apex,

R.A. 272°·4, Dec. +43°·1.

These regions are, however, not sufficiently numerous for a satisfactory determination of the general velocity ellipsoid; it is hoped to apply the present method to the Boss Stars.

It is a great pleasure to express my thanks to Dr. W. M. Smart for his valuable help in connection with this problem.

Trinity Hall, Cambridge:
1937 *January* 1.

THE DYNAMICS OF A STELLAR SYSTEM.

G. L. Clark, B.A.

(Communicated by W. M. Smart)

1. Some years ago Jeans † suggested a method of finding the general form of the spatio-velocity distribution function for a stellar system. If the stars are in a state of steady motion, the fundamental equation satisfied by this function is

$$u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + X\frac{\partial f}{\partial u} + Y\frac{\partial f}{\partial v} + Z\frac{\partial f}{\partial w} = 0, \qquad (1)$$

where (u, v, w) are the components of the velocity of a star at (x, y, z) and (X, Y, Z) the components of the force acting on the star.

Jeans showed that the general integral of this equation is an arbitrary function of the first integrals of motion of an individual star at (x, y, z). The first integrals are found by solving the Lagrangian subsidiary equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{du}{X} = \frac{dv}{Y} = \frac{dw}{Z}. \qquad (2)$$

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* *M.N.*, **87**, 137, 1927.

† *M.N.*, **76**, 79, 1915; *Astronomy and Cosmogony*, p. 356, 1928; *Problems of Cosmogony and Stellar Dynamics*, p. 232, 1919.

One such integral can be written down at once, namely, the integral of energy,

$$I_1 = \tfrac{1}{2}(u^2+v^2+w^2) - \phi = \text{const.},$$

and as Jeans points out it is only in special cases that other integrals exist. When the gravitational field is spherically symmetrical so that V is a function of r only, there are three other integrals,

$$\left.\begin{aligned} I_2 &= yw - zv = \text{const.},\\ I_3 &= zu - xw = \text{const.},\\ I_4 &= xv - yu = \text{const.},\end{aligned}\right\} \qquad (3)$$

which express the fact that the moments of momentum about the axes of co-ordinates remain constant. There are no other integrals except for special values of $V(r)$. As an example Jeans considers the case when $\phi = \alpha r^2$ where α is a constant. The additional integrals are

$$\left.\begin{aligned} I_5 &= u^2 - 2\alpha x^2 = \text{const.},\\ I_6 &= v^2 - 2\alpha y^2 = \text{const.},\\ I_7 &= w^2 - 2\alpha z^2 = \text{const.},\end{aligned}\right\} \qquad (4)$$

the corresponding motion being one in which each particle describes a continually repeated elliptic orbit about the centre. It should be observed that the constants of integration appearing in the above integrals are not all independent. To examine the relation between them it is only necessary to consider the two-dimensional case ; we do this in § 10.

Now previous to Jeans's paper Eddington * had investigated the values of a^2, b^2, c^2 when f had the special form

$$\frac{abc\sigma}{\pi^{3/2}} e^{-(a^2L^2+b^2M^2+c^2N^2)},$$

where L, M, N are the velocity components along the principal directions of an orthogonal system of co-ordinates. In his paper Eddington shows that Poisson's equation can only be satisfied by a system moving under its own gravitational attraction in the case of a spherical distribution. For a cylindrical or a prolate spheroidal distribution a controlling gravitational field has to be imposed. To avoid difficulty as to how the equilibrium of the controlling matter is maintained, Eddington dealt with a spherical distribution of matter. He further restricted the controlling distribution to be of uniform density. The potential is then given by

$$\phi = -E(1+r^2)$$

if the density of the controlling matter is $\dfrac{3E}{2\pi}$. This is actually the special value of the potential referred to above, giving a continually repeated elliptic orbit about the centre.

We therefore expect to obtain Eddington's values of a^2, b^2, c^2 by transforming the integrals to cylindrical and prolate co-ordinates, etc., and

* *M.N.*, **76**, 37, 1915.

combining them so that the resulting integrals are quadratic and containing no terms in LM, MN, NL. We do this in the case of general ellipsoidal co-ordinates (§ 7); for cylindrical and spheroidal systems we obtain the integrals directly from the Lagrangian subsidiary equations and then seek their physical interpretation. The equivalence of Jeans's method with that of Eddington is further demonstrated in § 9. The application to galactic rotation * is discussed in § 5.

In § 10 we consider the two-dimensional case: on examining an "invariant relation" we are led to criticise the statement of the theorem as given by Jeans, and conclude that, in general, it is necessary to carry out Eddington's analysis.

2. *Cylindrical Co-ordinates.*—If (ϖ, θ, z) be the cylindrical co-ordinates and (Π, Θ, Z) the corresponding components of velocity, the equations of motion are

$$\dot{\Pi} = \frac{\Theta^2}{\varpi} + \frac{\partial \phi}{\partial \varpi},$$

$$\dot{\Theta} = -\frac{R\Theta}{\varpi} + \frac{1}{\varpi}\frac{\partial \phi}{\partial \theta},$$

$$\dot{Z} = -\frac{\partial \phi}{\partial z}.$$

For a globular distribution of controlling matter

$$\begin{aligned}\phi &= -E(1 + \varpi^2 + z^2)\\ &= \phi_1(\varpi) + \phi_2(z)\end{aligned}$$

and

$$\frac{\partial \phi}{\partial \theta} = 0.$$

The subsidiary equations are

$$\frac{d\varpi}{\Pi} = \frac{d\theta}{\Theta/\varpi} = \frac{dz}{Z} = \frac{d\Pi}{\frac{\Theta^2}{\varpi} + \frac{\partial \phi}{\partial \varpi}} = \frac{d\Theta}{\frac{-\Pi\Theta}{\varpi}} = \frac{dZ}{\partial\phi/\partial z}. \tag{5}$$

We know the integrals of energy and angular momentum about the z-axis

$$\begin{aligned}I_1 &= \Pi^2 + \Theta^2 + Z^2 - 2\phi,\\ I_4 &= \varpi\Theta.\end{aligned}$$

Multiplying the numerators and denominators in (5) successively by

$$-Z\cos\theta, \quad Z\varpi\sin\theta - \Theta z\cos\theta, \quad \Pi\cos\theta - \Theta\sin\theta, \quad z\cos\theta,$$
$$-z\sin\theta \quad \text{and} \quad -\varpi\cos\theta,$$

we obtain the integral

$$I_2 = (\Pi z - Z\varpi)\cos\theta - \Theta z\sin\theta.$$

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* *Cf.* Eddington, *loc. cit.*, p. 48; Shiveshwarkar, *M.N.*, **96**, 755, 1936.

Also multiplying the numerators and denominators in (5) successively by

$$Z\sin\theta, \quad -\Pi z\cos\theta+\Theta z\sin\theta, \quad -\Pi\sin\theta-\Theta\cos\theta, \quad -z\sin\theta,$$
$$-z\cos\theta \quad \text{and} \quad \varpi\sin\theta,$$

we obtain

$$I_3=(Z\varpi-\Pi z)\sin\theta-\Theta z\cos\theta.$$

It is evident that I_2 and I_3 are the integrals of the moment of momentum about the x and y axes respectively.

Since ϕ is separable in ϖ and z we are able to obtain a further integral: the energy-integral in the z-co-ordinate. This integral is

$$I_7=Z^2-2\phi_2(z)$$

and is obtained by multiplying the third and last quantities of (5) by $\frac{\partial\phi_2}{\partial z}$ and Z respectively.

For a quadratic distribution of velocity of the form we are at present considering

$$f=f(I_1+k_1I_7+k_2I_4^2)$$
$$=f(\Pi^2+(1+k_2\varpi^2)\Theta^2+(1+k_1)Z^2-2\phi-2k_1\phi_2).$$

Taking $f(x)=e^{-x}$ we see that so far as the velocity is concerned

$$f=e^{-a^2\Pi^2-b^2Z^2-c^2\Theta^2},$$

where

$$a^2=\text{const.}, \qquad b^2=\text{const.}, \qquad c^2=a^2(1+\varpi^2),$$

on suitably choosing the unit of length. These are the values obtained by Eddington.*

We may remark that in the case considered by Jeans when only two integrals are known, only a special solution is obtained, namely

$$a^2=b^2=\text{const.}$$

By taking a special form of ϕ such that

$$\phi=\phi_1(\varpi)+\phi_2(z)$$

we obtain a third integral, and hence arrive at the more general result

$$a^2=\text{const.}, \qquad b^2=\text{const.}$$

3. *Prolate Spheroidal Distribution.*—The prolate spheroidal co-ordinates are given by the equations

$$x=\sinh\xi\sin\eta\cos\gamma, \qquad y=\sinh\xi\sin\eta\sin\gamma, \qquad z=\cosh\xi\cos\eta.$$

The elements of length in the principal directions are

$$Pd\xi, \qquad Qd\eta, \qquad Rd\gamma,$$

where

$$P^2=Q^2=\cosh^2\xi-\cos^2\eta, \qquad R^2=\sinh^2\xi\sin^2\eta.$$

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* *Loc. cit.*, p. 45, § 6 (*b*).

The velocity components (LMN) along the principal directions are

$$L = P\dot{\xi}, \qquad M = Q\dot{\eta}, \qquad N = R\dot{\gamma}.$$

We have for the kinetic and potential energies

$$T = \tfrac{1}{2}(L^2 + M^2 + N^2),$$
$$V = -\phi.$$

Lagrange's equation gives

$$\frac{DL}{Dt} = \frac{1}{2P}\frac{\partial}{\partial \xi}(L^2 + M^2 + N^2) + \frac{1}{P}\frac{\partial \phi}{\partial \xi},$$

i.e.

$$\begin{aligned}\dot{L} &= -\frac{\partial L}{\partial \xi}\dot{\xi} - \frac{\partial L}{\partial \eta}\dot{\eta} + \frac{L}{P}\frac{\partial L}{\partial \xi} + \frac{M}{P}\frac{\partial M}{\partial \xi} + \frac{N}{P}\frac{\partial N}{\partial \xi} + \frac{1}{P}\frac{\partial \phi}{\partial \xi} \\ &= -\frac{\partial P}{\partial \eta}\frac{LM}{P^2} + \frac{\partial P}{\partial \xi}\frac{M^2}{P^2} + \frac{\partial R}{\partial \xi}\frac{N^2}{PR} + \frac{1}{P}\frac{\partial \phi}{\partial \xi} \\ &= -\cos\eta \sin\eta \frac{LM}{P^3} + \sinh\xi \cosh\xi \frac{M^2}{P^3} + \coth\xi \frac{N^2}{P} + \frac{1}{P}\frac{\partial \phi}{\partial \xi}. \end{aligned} \tag{6}$$

Similarly, we have on writing P for Q,

$$\frac{DM}{Dt} = \frac{1}{2P}\frac{\partial}{\partial \eta}(L^2 + M^2 + N^2) + \frac{1}{P}\frac{\partial \phi}{\partial \eta},$$

whence

$$\dot{M} = -\sinh\xi \cosh\xi \frac{LM}{P^3} + \sin\eta \cos\eta \frac{L^2}{P^3} + \cot\eta \frac{N^2}{P} + \frac{1}{P}\frac{\partial \phi}{\partial \eta}. \tag{7}$$

The third Lagrange's equation

$$\frac{DN}{Dt} = \frac{1}{R}\frac{\partial \phi}{\partial \gamma} = 0$$

gives

$$\begin{aligned}\dot{N} &= -\frac{\partial R}{\partial \xi}\frac{LN}{PR} - \frac{\partial R}{\partial \eta}\frac{MN}{PR} \\ &= -\coth\xi \frac{LN}{P} - \cot\eta \frac{MN}{P}. \end{aligned} \tag{8}$$

The fundamental equation for a steady state is

$$\frac{\partial f}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial f}{\partial \eta}\frac{d\eta}{dt} + \frac{\partial f}{\partial \gamma}\frac{d\gamma}{dt} + \frac{\partial f}{\partial L}\frac{dL}{dt} + \frac{\partial f}{\partial M}\cdot\frac{dM}{dt} + \frac{\partial f}{\partial N}\cdot\frac{dN}{dt} = 0,$$

i.e.

$$\frac{L}{P}\frac{\partial f}{\partial \xi} + \frac{M}{P}\frac{\partial f}{\partial \eta} + \frac{N}{R}\frac{\partial f}{\partial \gamma} + \dot{L}\frac{\partial f}{\partial L}\dot{M} + \frac{\partial f}{\partial M} + \dot{N}\frac{\partial f}{\partial N} = 0.$$

The subsidiary equations are

$$\frac{d\xi}{L/P} = \frac{d\eta}{M/P} = \frac{d\gamma}{N/R} = \frac{dL}{\dot{L}} = \frac{dM}{\dot{M}} = \frac{dN}{\dot{N}}. \tag{9}$$

Multiplying the denominators and numerators successively by

$$-\frac{\partial\phi}{\partial\xi}, \qquad -\frac{\partial\phi}{\partial\eta}, \qquad 0, \qquad L, \qquad M, \qquad N,$$

we obtain the integral of energy

$$I_1 = L^2 + M^2 + N^2 - 2\phi. \tag{10}$$

Again multiplying the numerators and denominators of the first, second and last quantities in (9) by

$$\cosh\xi \sin\eta N, \qquad \sinh\xi\cos\eta N, \qquad \sinh\xi\sin\eta,$$

we obtain the integral of momentum about the z-axis,

$$I_4 = \sinh\xi\sin\eta N. \tag{11}$$

Multiplying the numerators and denominators in (9) successively by

$$\begin{aligned}&\sinh\xi\cosh\xi(M^2+N^2) + 2E\cosh\xi\sinh\xi\cos^2\eta,\\ &-\cos\eta\sin\eta(L^2+N^2) - 2E\cosh^2\xi\cos\eta\sin\eta,\\ &\cos^2\eta L,\\ &\cosh^2\xi M,\end{aligned}$$

and

$$(\sinh^2\xi + \cos^2\eta)N,$$

we obtain the integral

$$I' = \cos^2\eta L^2 + \cosh^2\xi M^2 + (\sinh^2\xi + \cos^2\eta)N^2 - 2F, \tag{12}$$

where

$$F = -E\cosh^2\xi\cos^2\eta.$$

To obtain the physical significance of this integral we transform to cylindrical co-ordinates by the relations

$$\varpi = \sinh\xi\sin\eta, \qquad z = \cosh\xi\cos\eta, \qquad \theta = \gamma.$$

From these we have

$$\Pi = \cosh\xi\sin\eta\frac{L}{P} + \sinh\xi\cos\eta\frac{M}{P},$$

$$Z = \sinh\xi\cos\eta\frac{L}{P} - \cosh\xi\sin\eta\frac{M}{P},$$

$$\Theta = N,$$

whence

$$\Pi z - Z\varpi = \cos\eta\sin\eta\frac{L}{P} + \sinh\xi\cosh\xi\frac{M}{P}.$$

Therefore

$$\begin{aligned}(\Pi z - Z\varpi)^2 + Z^2 &= \cos^2\eta(\sinh^2\xi + \sin^2\eta)\frac{L^2}{P^2} + \cosh^2\xi(\sinh^2\xi + \sin^2\eta)\frac{L^2}{P^2}\\ &= \cos^2\eta L^2 + \cosh^2\xi M^2.\end{aligned}$$

Also

$$\begin{aligned}(\varpi^2 + z^2)N^2 &= (\sinh^2\xi\sin^2\eta + \cosh^2\xi\cos^2\eta)N^2\\ &= (\sinh^2\xi + \cos^2\eta)N^2.\end{aligned}$$

The total angular momentum is given by

$$(\varpi^2+z^2)N^2+(\Pi z-Z\varpi)^2.$$

Accordingly we identify (12) as the sum of two independent integrals, one being the integral of total angular momentum and the other the energy-integral in the z co-ordinate.

The distribution function is obtained from (10), (11), (12) and is

$$\begin{aligned} f&=f(AI_1+BI'+CI_4)\\ &=f(a^2L^2+b^2M^2+c^2N^2-2BF-2A\phi), \end{aligned}$$

where

$$\begin{aligned} a^2&=A+B\cos^2\eta,\\ b^2&=A+B\cosh^2\eta,\\ c^2&=A+B(\sinh^2\xi+\cos^2\eta)+C\sinh^2\xi\sin^2\eta. \end{aligned}$$

These are the values of a^2, b^2, c^2 obtained by Eddington.

4. *Expressions for the Star Denisty and Mean Velocity.*—Taking $f(x)=e^{-x}$ the star density is given by

$$\begin{aligned} \nu&=e^{2A\phi+2BF}\iiint_{-\infty}^{\infty}e^{-a^2L^2-b^2M^2-c^2N^2}dLdMdN\\ &=\frac{\pi^{3/2}}{abc}e^{2A\phi+2BF}\\ &=\text{const.}\ \frac{\pi^{\frac{3}{2}}}{abc}e^{-2E\{A(\cosh^2\xi+\cos^2\eta)+B\cosh^2\xi\cos^2\eta\}}. \end{aligned}$$

If a bar denotes the mean values, we have

$$\nu\overline{L^2}=\frac{\pi^{3/2}}{2a^3bc}e^{(2A\phi+2BF)}$$

and similar results for $\overline{M^2}$ and $\overline{N^2}$.

We therefore have

$$\overline{L^2}:\overline{M^2}:\overline{N^2}=\frac{1}{a^2}:\frac{1}{b^2}:\frac{1}{c^2}.$$

5. *Galactic Rotation.*—Considering only the integrals I_1, I_4, and I' given by (10), (11) and (12) we see that a more general form for f is

$$\begin{aligned} f&=f(AI_1+CI_4^2+BI'+2kI_4)\\ &=f(a^2L^2+b^2M^2+c^2N^2+2kN\sinh\xi\sin\eta-2V_1)\\ &=f(a^2L^2+b^2M^2+c^2(N-N_0)^2-2G), \end{aligned}$$

where

$$\begin{aligned} 2V_1&=2A\phi+2BF,\\ N_0&=-\frac{k}{c^2}\sinh\xi\sin\eta,\\ 2G&=2V_1+\frac{k^2}{c^2}\sinh^2\xi\sin^2\eta. \end{aligned}\qquad(13)$$

Put

$$aL=L_1,\qquad bM=M_1,\qquad c(N-N_0)=N_1,$$

then

$$f = f(c^2 - 2G),$$

where

$$c^2 = L_1^2 + M_1^2 + N_1^2.$$

The star density is given by

$$\nu = \iiint_{-\infty}^{\infty} f dL dM dN$$

$$= \frac{4\pi}{abc}\int_0^{\infty} f(c^2 - 2G)c^2 dc. \tag{14}$$

If

$$p = \nu\overline{L^2}, \qquad q = \nu\overline{M^2}, \qquad r = \nu\overline{N^2}, \qquad r' = \nu\overline{(N - N_0)^2},$$

then

$$p = \frac{4\pi}{3a^3bc}\int_0^{\infty} f(c^2 - 2G)c^4 dc,$$

$$q = \frac{4\pi}{3ab^3c}\int_0^{\infty} f(c^2 - 2G)c^4 dc,$$

$$r' = \frac{4\pi}{3abc^3}\int_0^{\infty} f(c^2 - 2G)c^4 dc,$$

$$r = \nu N_0^2 + r'.$$

Hence

$$q = \frac{a^2}{b^2}p,$$

$$r' = \frac{a^2}{c^2}p,$$

$$r = \nu N_0^2 + \frac{a^2}{c^2}p,$$

or

$$r = \nu N_0^2 + \frac{b^2}{c^2}q.$$

These give

$$\overline{N^2} = N_0^2 + \overline{(N - N_0)^2}.$$

It is evident that N_0 is the differential galactic rotation.

Taking $f(x) = e^{-x}$ we get

$$\overline{L^2} : \overline{M^2} : \overline{(N - N_0)^2} = \frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2}.$$

The angular velocity is given by

$$\omega = \frac{N_0}{r} = -\frac{k}{c^2} = \frac{-k}{A + B\sinh^2\xi + B\cos^2\eta + C\sinh^2\xi\sin^2\eta}.$$

For the galactic plane

$$\omega = \frac{N_0}{r} = \frac{-k}{A + (B + C)r^2}. \tag{15}$$

On referring to (13) and (14) we see that the rotation increases the density by a factor

$$e^{+\frac{k^2}{c^2}\sinh^2\xi\sin^2\eta} = e^{+r^2k^2/c^2},$$

which for the galactic plane is

$$e^{+\frac{r^2k^2}{A+(B+C)r^2}}.$$

The results are identical with those obtained by Eddington. It is also of interest to note that by using a method due to Shiveshwarkar * we can derive the relation

$$\lambda^{-1} = \sqrt{1 + A_0/\omega},$$

where A_0 is Oort's constant (generally written A) and

$$\lambda = \frac{c}{a} = \sqrt{1 + \frac{B+C}{A}r^2}.$$

For from (15) we have

$$N_0 r = \frac{-kr^2}{A+(B+C)r^2}.$$

Differentiating logarithmically with respect to r,

$$\frac{1}{N_0}\frac{dN_0}{dr} + \frac{1}{r} = \frac{2}{r\left(1+\frac{B+C}{A}r^2\right)} = \frac{2}{r\lambda^2}.$$

Also

$$A_0 = \tfrac{1}{2}\left(\frac{dN_0}{dr} - \frac{N_0}{r}\right) \quad \text{and} \quad \omega = \frac{N_0}{r}.$$

Therefore

$$\lambda^{-1} = \sqrt{1 + A_0/\omega}.$$

In the case discussed by Shiveshwarkar $B = 0$.

6. *Oblate Spheroidal Distribution.*—As this case is very similar to the previous one we will not work out the problem in detail. We merely quote a few results.

The equations connecting oblate co-ordinates (ξ, η, γ) with Cartesian co-ordinates (x, y, z) are

$$x = \cosh\xi\cos\eta\cos\gamma, \qquad y = \cosh\xi\cos\eta\sin\gamma, \qquad z = \sinh\xi\sin\eta,$$

and

$$P^2 = Q^2 = \cosh^2\xi - \cos^2\eta, \qquad R = \cosh\xi\cos\eta.$$

The integrals corresponding to I_1, I_4, I' of the previous case are

$$I_1 = L^2 + M^2 + N^2 - 2\phi,$$
$$I_4 = \cosh\xi\cos\eta N,$$
$$I' = \cos^2\eta L^2 + \cosh^2\xi M^2 + (\cosh^2\xi + \cos^2\eta)N^2 - 2F,$$

where F has the same value as before.

* *M.N.*, **96**, 757, 1936.

The general form of the distribution function of the type required is

$$f = f(-2BF - 2A\phi + a^2L^2 + b^2M^2 + c^2N^2),$$

where

$$a^2 = A + B\cos^2\eta,$$
$$b^2 = A + B\cosh^2\xi,$$
$$c^2 = A + B\cosh^2\xi + B\cos^2\eta + C\cosh^2\xi\cos^2\eta.$$

These values of a^2, b^2, c^2 are given by Eddington, § 6 (*e*).

7. *General Ellipsoidal Co-ordinates.*—Following Eddington we take the squares of the semi-axes of the confocals to be

$$\lambda^2,\ \lambda^2-\beta^2,\ \lambda^2-\gamma^2; \qquad \mu^2,\ \mu^2-\beta^2,\ \mu^2-\gamma^2; \qquad \nu^2,\ \nu^2-\beta^2,\ \nu^2-\gamma^2.$$

The Cartesian co-ordinates are given by

$$x^2 = \frac{\lambda^2\mu^2\nu^2}{\beta^2\gamma^2}, \qquad y^2 = \frac{(\lambda^2-\beta^2)(\mu^2-\beta^2)(\nu^2-\beta^2)}{\beta^2(\beta^2-\gamma^2)}, \qquad z^2 = \frac{(\lambda^2-\gamma^2)(\mu^2-\gamma^2)(\gamma^2-\nu^2)}{\gamma^2(\beta^2-\gamma^2)}$$

and

$$P^2 = \frac{(\lambda^2-\mu^2)(\lambda^2-\nu^2)}{(\lambda^2-\beta^2)(\lambda^2-\gamma^2)}, \qquad Q^2 = \frac{(\mu^2-\nu^2)(\lambda^2-\mu^2)}{(\mu^2-\beta^2)(\gamma^2-\mu^2)}, \qquad R^2 = \frac{(\lambda^2-\nu^2)(\mu^2-\nu^2)}{(\beta^2-\nu^2)(\gamma^2-\nu^2)}.$$

The Cartesian components of velocity (u, v, w) are

$$u = \frac{1}{\beta\gamma}\left\{\mu\nu\frac{L}{P} + \nu\lambda\frac{M}{Q} + \lambda\mu\frac{N}{R}\right\}, \tag{16}$$

$$v = \frac{\sqrt{(\lambda^2-\beta^2)(\mu^2-\beta^2)(\nu^2-\beta^2)}}{\beta\sqrt{\beta^2-\gamma^2}}\left\{\frac{\lambda}{\lambda^2-\beta^2}\frac{L}{P} + \frac{\mu}{\mu^2-\beta^2}\frac{M}{Q} + \frac{\nu}{\nu^2-\beta^2}\frac{N}{R}\right\}, \tag{17}$$

$$w = \frac{\sqrt{(\lambda^2-\gamma^2)(\mu^2-\gamma^2)(\gamma^2-\nu^2)}}{\gamma\sqrt{\beta^2-\gamma^2}}\left\{\frac{\lambda}{\lambda^2-\gamma^2}\frac{L}{P} + \frac{\mu}{\mu^2-\gamma^2}\frac{M}{Q} + \frac{\nu N}{\nu^2-\gamma^2 R}\right\}, \tag{18}$$

whence

$$vz - wy = \frac{\sqrt{(\lambda^2-\beta^2)(\lambda^2-\gamma^2)(\mu^2-\beta^2)(\mu^2-\gamma^2)(\nu^2-\beta^2)(\gamma^2-\nu^2)}}{\beta\gamma}$$
$$\times\left\{\frac{\lambda}{(\lambda^2-\beta^2)(\lambda^2-\gamma^2)}\frac{L}{P} + \frac{\mu}{(\mu^2-\beta^2)(\mu^2-\gamma^2)}\frac{M}{Q} + \frac{\nu}{(\nu^2-\beta^2)(\nu^2-\gamma^2)}\frac{N}{R}\right\}, \tag{19}$$

$$wx - uz = \frac{\lambda\mu\nu}{\beta\sqrt{\beta^2-\gamma^2}}\sqrt{(\lambda^2-\gamma^2)(\mu^2-\gamma^2)(\gamma^2-\nu^2)}$$
$$\left\{\frac{1}{\lambda(\lambda^2-\gamma^2)}\frac{L}{P} + \frac{1}{\mu(\mu^2-\gamma^2)}\frac{M}{Q} + \frac{1}{\nu(\nu^2-\gamma^2)}\frac{N}{R}\right\}, \tag{20}$$

$$vx - uy = \frac{\lambda\mu\nu}{\gamma\sqrt{\beta^2-\gamma^2}}\sqrt{(\lambda^2-\beta^2)(\mu^2-\beta^2)(\nu^2-\beta^2)}$$
$$\left\{\frac{1}{\lambda(\lambda^2-\beta^2)}\frac{L}{P} + \frac{1}{\mu(\mu^2-\beta^2)}\frac{M}{Q} + \frac{1}{\nu(\nu^2-\beta^2)}\frac{N}{R}\right\}. \tag{21}$$

The integrals are given by (3) and (4) on writing $\alpha = -E$, namely:

$$\begin{aligned} I_2 &= (wy - vz), & I_5 &= u^2 + 2Ex^2, \\ I_3 &= (uz - wx), & I_6 &= v^2 + 2Ey^2, \\ I_4 &= (vx - uy), & I_7 &= w^2 + 2Ez^2. \end{aligned}$$

We require to find combinations of these integrals which are of the form

$$g_1L^2 + g_2M^2 + g_3N^2 + F = \text{const.},$$

where g_1, g_2, g_3 and F can be functions of $(\lambda\mu\nu)$.

We first form the expression for

$$\beta^2 I_2{}^2 + (\beta^2 - \gamma^2) I_4{}^2.$$

The coefficient of $2\dfrac{LM}{PQ}$ is

$$\lambda\mu(\nu^2 - \beta^2).$$

The coefficient of L^2/P^2 is

$$\frac{(\nu^2 - \beta^2)(\mu^2 - \beta^2)}{\gamma^2(\lambda^2 - \beta^2)} \left\{ \frac{(\mu^2 - \gamma^2)(\gamma^2 - \nu^2)}{(\lambda^2 - \gamma^2)} \lambda^2 + \mu^2\nu^2 \right\}$$

$$= -(\nu^2 - \beta^2)(\mu^2 - \beta^2)\left(P^2 - \frac{\lambda^2}{\lambda^2 - \beta^2} \right).$$

Hence we obtain

$$\begin{aligned} &\beta^2(\beta^2 - \gamma^2) I_6 - \beta^2 I_2{}^2 - (\beta^2 - \gamma^2) I_4{}^2 \\ &\quad = (\nu^2 - \beta^2)(\mu^2 - \beta^2) L^2 + + + 2E(\lambda^2 - \beta^2)(\mu^2 - \beta^2)(\nu^2 - \beta^2). \end{aligned} \tag{22}$$

Again in the expression for

$$\beta^2 I_3{}^2 + \gamma^2 I_4{}^2$$

the coefficient of $2\lambda\mu\dfrac{LM}{PQ}$ is

$$\frac{\nu^2\{(\nu^2 - \beta^2) + (\gamma^2 - \nu^2)\}}{\beta^2 - \gamma^2} = -\nu^2,$$

and the coefficient of L^2/P^2 is

$$\begin{aligned} &\frac{\mu^2\nu^2}{(\beta^2 - \gamma^2)(\lambda^2 - \beta^2)(\lambda^2 - \gamma^2)} \{(\lambda^2 - \gamma^2)(\mu^2 - \beta^2)(\nu^2 - \beta^2) + (\lambda^2 - \beta^2)(\mu^2 - \gamma^2)(\gamma^2 - \nu^2)\} \\ &\quad = \frac{\mu^2\nu^2}{(\lambda^2 - \beta^2)(\lambda^2 - \gamma^2)} \{ -\lambda^2(\mu^2 + \nu^2) + \lambda^2(\beta^2 + \gamma^2) + \mu^2\nu^2 - \beta^2\gamma^2 \} \\ &\quad = \mu^2\nu^2(-1 + P^2). \end{aligned}$$

Using (16) we then get

$$\beta^2\gamma^2 I_5 + \beta^2 I_3{}^2 + \gamma^2 I_4{}^2 = \mu^2\nu^2 L^2 + \nu^2\lambda^2 M^2 + \lambda^2\mu^2 N^2 + 2E\lambda^2\mu^2\nu^2 = \text{const.} \tag{23}$$

The integral (22) can be written

$$\begin{aligned} &(\nu^2\mu^2 L^2 + \lambda^2\nu^2 M^2 + \mu^2\lambda^2 N^2) - \beta^2\{(\mu^2 + \nu^2)L^2 + (\nu^2 + \lambda^2)M^2 + (\lambda^2 + \mu^2)N^2\} \\ &\qquad + \beta^4\{L^2 + M^2 + N^2\} + 2E(\lambda^2 - \beta^2)(\mu^2 - \beta^2)(\nu^2 - \beta^2) = \text{const.} \end{aligned}$$

The expression in the second bracket is a part of the integral (23) and that in the third bracket a part of the energy integral. We therefore see that a third independent integral is

$$(\mu^2+\nu^2)L^2+(\nu^2+\lambda^2)M^2+(\lambda^2+\mu^2)N^2+F(\lambda\mu\nu),$$

where the function $F(\lambda\mu\nu)$ can be determined, but the actual form is of no interest.

We cannot combine the integrals I_2 . . . I_7 to give any other independent integrals of the form required. It therefore follows that the distribution function is of the form

$$f=(a^2L^2+b^2M^2+c^2N^2),$$

where

$$\begin{aligned} a^2 &= A+B(\mu^2+\nu^2)+c\mu^2\nu^2,\\ b^2 &= A+B(\nu^2+\lambda^2)+c\nu^2\lambda^2,\\ c^2 &= A+B(\lambda^2+\mu^2)+c\lambda^2\mu^2. \end{aligned}$$

8. Before considering formally the equivalence of the two theories we point out in this paragraph that in the special cases of rectangular, cylindrical and spherical co-ordinates the conditions imposed by Eddington on the potential ϕ are sufficient to give the three necessary integrals.

The conditions are :

$$\begin{aligned} \frac{\partial^2}{\partial\lambda\partial\mu}\{(a^2-b^2)\phi\} &= 0,\\ \frac{\partial^2}{\partial\mu\partial\nu}\{(b^2-c^2)\phi\} &= 0,\\ \frac{\partial^2}{\partial\nu\partial\lambda}\{(c^2-a^2)\phi\} &= 0, \end{aligned}$$

where $(\lambda\mu\nu)$ are general orthogonal co-ordinates.

(*a*) Cartesian Co-ordinates : $a^2=\text{const.}$, $b^2=\text{const.}$, $c^2=\text{const.}$ Hence

$$\frac{\partial^2\phi}{\partial x\partial y}=0, \qquad \frac{\partial^2\phi}{\partial y\partial z}=0, \qquad \frac{\partial^2\phi}{\partial z\partial x}=0,$$

whence

$$\phi=\phi_1(x)+\phi_2(y)+\phi_3(z).$$

This gives energy-integrals in each co-ordinate.

(*b*) Cylindrical Co-ordinates : $a^2=\text{const.}$, $b^2=\text{const.}$, $c^2=a^2(1+\varpi^2)$. Hence

$$\frac{\partial^2\phi}{\partial\varpi\partial z}=0, \tag{i}$$

$$\frac{\partial^2}{\partial z\partial\theta}\{[b^2-a^2(1+\varpi^2)]\phi\}=0, \tag{ii}$$

$$\frac{\partial^2}{\partial\varpi\partial\theta}\{\varpi^2\phi\}=0. \tag{iii}$$

From (i) we have $\phi = f(\varpi\theta) + g(z\theta)$.
Substituting in (ii) gives $g(z)$ independent of θ.
Condition (iii) then gives $f(\varpi)$ independent of θ.
Hence

$$\phi = f(\varpi) + g(z),$$

and is separable in z and independent of θ.

The three integrals are therefore

(α) Energy integral.
(β) Energy integral in the z co-ordinate.
(γ) Angular momentum integral about z-axis.

(*c*) Spherical Co-ordinates : $a^2 = \text{const.}$, $b^2 = a^2(1 + r^2)$, $c^2 = a^2(1 + r^2 + Cr^2 \sin^2\theta)$. The conditions on ϕ are (writing V for ϕ) :

$$\frac{\partial^2}{\partial r \partial \theta}(r^2 V) = 0, \tag{i}$$

$$\frac{\partial}{\partial r \partial \phi}\{(r^2 + Cr^2 \sin^2\theta)V\} = 0, \tag{ii}$$

$$\frac{\partial^2}{\partial \theta \partial \phi}\{r^2 \sin^2\theta V\} = 0. \tag{iii}$$

From (i)

$$V = \frac{f(r\phi) + g(\theta\phi)}{r^2}.$$

Substituting in (ii)

$$\frac{\partial}{\partial r \partial \phi}\{(1 + C\sin^2\theta) f(r\phi) + g(\theta\phi)\} = 0,$$

i.e.

$$\frac{\partial f}{\partial \phi} = 0.$$

Therefore

$$V = \frac{f(r) + g(\theta\phi)}{r^2}.$$

Substituting in (iii) we now get

$$\frac{\partial}{\partial \theta}\left(\sin^2\theta \frac{\partial g}{\partial \phi}\right) = 0,$$

whence g is independent of ϕ.

The potential is therefore given by

$$\phi = \frac{f(r) + g(\theta)}{r^2}.$$

Eddington has pointed out that the only practical solution is when $g(\theta)$ is a constant. The three integrals are then

(α) The energy integral.
(β) The angular momentum integral about the z-axis.
(γ) The integral of angular momentum about the origin.

9. We now consider a steady stellar system in general orthogonal co-ordinates. By obtaining conditions necessary for the Lagrangian subsidiary equations to admit of integrals other than that of energy we show the equivalence of the two methods in the case when the distribution function is of the form

$$f = f(a^2L^2 + b^2M^2 + c^2N^2).$$

The equations of motion are

$$P\dot{L} = \frac{1}{2}\frac{\partial Q^2}{\partial \lambda}\frac{M^2}{Q^2} + \frac{1}{2}\frac{\partial R^2}{\partial \lambda}\frac{N^2}{R^2} - \frac{1}{2}\frac{\partial P^2}{\partial \mu}\frac{LM}{PQ} - \frac{1}{2}\frac{\partial P^2}{\partial \nu}\frac{LN}{PR} + \frac{\partial \phi}{\partial \lambda}, \tag{24}$$

and two similar equations.

The fundamental equation is

$$\frac{\partial f}{\partial \lambda}\frac{L}{P} + \frac{\partial f}{\partial \mu}\frac{M}{Q} + \frac{\partial f}{\partial \nu}\frac{N}{R} + \frac{\partial f}{\partial L}\dot{L} + \frac{\partial f}{\partial M}\dot{M} + \frac{\partial f}{\partial N}\dot{N} = 0.$$

The solution is found by determining the integrals of the subsidiary equations

$$\frac{d\lambda}{L/P} = \frac{d\mu}{M/Q} = \frac{d\nu}{N/R} = \frac{dL}{\dot{L}} = \frac{dM}{\dot{M}} = \frac{dN}{\dot{N}}.$$

Suppose there are n such integrals $I_1 \ldots I_n$, from these we can form integrals $\mathcal{J}_i$ which are of the form

$$\mathcal{J}_i = g_iL^2 + h_iM^2 + k_iN^2 + 2F_i$$

where g_i, h_i, k_i, F_i are functions of $(\lambda\mu\nu)$. From § 3 it is clear that all the integrals I_i may not be used. Suppose there are k such integrals $\mathcal{J}_i$: then the most general form will be

$$\begin{aligned}\mathcal{J} &= gL^2 + hM^2 + kN^2 + 2F \\ &= a^2L^2 + b^2N^2 + c^2N^2 + 2F,\end{aligned}$$

where

$$\begin{aligned}a^2 &= g = \sum_{i=1}^{k} g_i, \\ b^2 &= h = \Sigma h_i, \\ c^2 &= k = \Sigma k_i, \\ F &= \Sigma F_i.\end{aligned}$$

We now establish two theorems relating to the existence of the solutions $\mathcal{J}_i$ and the number of such solutions.

Theorem 1.—The conditions that the subsidiary equations should possess integrals $\mathcal{J}_i$ (other than the integral of energy) are that

(1) P, Q, R refer to generalised confocal co-ordinates.

(2) ϕ satisfies certain conditions which can be stated.

Proof.—Differentiating the expression

$$a^2L^2 + b^2M^2 + c^2N^2 + 2F = \text{const.}$$

we get

$$\begin{aligned}&2a^2LdL+2b^2MdM+2c^2NdN\\&+\left(\frac{\partial a^2}{\partial\lambda}L^2+\frac{\partial b^2}{\partial\lambda}M^2+\frac{\partial c^2}{\partial\lambda}N^2\right)d\lambda\\&+\left(\frac{\partial a^2}{\partial\mu}L^2+\frac{\partial b^2}{\partial\mu}M^2+\frac{\partial c^2}{\partial\mu}N^2\right)d\mu\\&+\left(\frac{\partial a^2}{\partial\nu}L^2+\frac{\partial b^2}{\partial\nu}M^2+\frac{\partial c^2}{\partial\nu}N^2\right)d\nu\\&+2\frac{\partial F}{\partial\lambda}d\lambda+2\frac{\partial F}{\partial\mu}d\mu+2\frac{\partial F}{\partial\nu}d\nu=0.\end{aligned}$$

Substituting $\frac{L}{P}$ for $d\lambda$ and the expression (24) for $\dot{L}$ in place of dL, etc., we obtain an identity, from which, on equating the coefficients of the various powers of L, M, N to zero, we derive the set of equations

$$\left.\begin{array}{lll}\dfrac{\partial a^2}{\partial\lambda}=0, & \dfrac{\partial b^2}{\partial\mu}=0, & \dfrac{\partial c^2}{\partial\nu}=0,\\ \dfrac{\partial}{\partial\mu}\left(\dfrac{b^2-c^2}{R^2}\right)=0, & \dfrac{\partial}{\partial\nu}\left(\dfrac{c^2-a^2}{R^2}\right)=0, & \dfrac{\partial}{\partial\lambda}\left(\dfrac{a^2-b^2}{Q^2}\right)=0,\\ \dfrac{\partial}{\partial\nu}\left(\dfrac{b^2-c^2}{Q^2}\right)=0, & \dfrac{\partial}{\partial\lambda}\left(\dfrac{c^2-a^2}{R^2}\right)=0, & \dfrac{\partial}{\partial\mu}\left(\dfrac{a^2-b^2}{P^2}\right)=0,\\ \dfrac{\partial F}{\partial\lambda}=-a^2\dfrac{\partial\phi}{\partial\lambda}, & \dfrac{\partial F}{\partial\mu}=-b^2\dfrac{\partial\phi}{\partial\lambda}, & \dfrac{\partial F}{\partial\nu}=-c^2\dfrac{\partial\phi}{\partial\nu}.\end{array}\right\}\qquad(25)$$

Equations (25) are identical with those obtained by Eddington. His proof that P, Q, R refer to confocal quadrics then follows. It may be noticed that the problem in the two-dimensional case had been treated earlier by Whittaker in his *Analytical Dynamics*; * the result is there stated in the following terms: "The only cases of the motion of a particle in a plane under the action of conservative forces, which possess an integral quadratic in the velocities other than the integral of energy, are those for which the potential energy has the form

$$V=\frac{f(\alpha)-\phi(\beta)}{\alpha^2-\beta^2}$$

where α and β are the parameters of confocal ellipses and hyperbolas."

The last row of (25) gives Eddington's conditions for ϕ, viz.

$$\frac{\partial^2}{\partial\lambda\partial\mu}\{(a^2-b^2)\phi\}=0$$

and two similar equations.

Theorem I is therefore established for a system in a steady state. The restriction on P, Q, R holds for a system in a non-steady state. In this case

* *Analytical Dynamics*, chap. xii, p. 334 (second edition, 1917).

there are extra terms $\frac{\partial P}{\partial t}\frac{L}{P}$, etc., in the equations of motion. The integral $\mathcal{J}$ now takes the form

$$\mathcal{J} = gL^2 + hM^2 + kN^2 + g_1L + h_1M + k_1N + 2F,$$

where g, h, k, g_1, h_1, k_1, F are functions of λ, μ, ν and t. On forming the identity in the manner indicated above we find that the terms cubic in the velocities remain unaltered. The necessity for P, Q, R to refer to confocal quadrics is then evident. The last row of (25), however, contains extra terms.

Theorem II.—The conditions of Theorem I being satisfied there are two other independent integrals (*i.e.* there are three independent integrals in all) and these can be determined.

For from equations (25) the values of a^2, b^2, c^2 can be obtained. It is found that these contain three arbitrary constants and therefore there are only three independent integrals of the type $\mathcal{J}_i$. We consider here only the most general case, namely, in generalised ellipsoidal co-ordinates.

The squares of the semi-axes of the confocals through (λ, μ, ν) being

$$\lambda^2,\ \lambda^2-\beta^2,\ \lambda^2-\gamma^2; \qquad \mu^2,\ \mu^2-\beta^2,\ \mu^2-\gamma^2; \qquad \nu^2,\ \nu^2-\beta^2,\ \nu^2-\gamma^2.$$

$$P^2 = \frac{(\lambda^2-\mu^2)(\lambda^2-\nu^2)}{(\lambda^2-\beta^2)(\lambda^2-\gamma^2)}, \qquad Q^2 = \frac{(\mu^2-\nu^2)(\lambda^2-\mu^2)}{(\mu^2-\beta^2)(\gamma^2-\mu^2)}, \qquad R^2 = \frac{(\lambda^2-\nu^2)(\mu^2-\nu^2)}{(\beta^2-\nu^2)(\gamma^2-\nu^2)}.$$

Eddington has worked out the values of a^2, b^2, c^2 for these values of P, Q, R, his results being

$$\begin{aligned} a^2 &= A + B(\mu^2+\nu^2) + C\mu^2\nu^2, \\ b^2 &= A + B(\nu^2+\lambda^2) + C\nu^2\lambda^2, \\ c^2 &= A + B(\lambda^2+\mu^2) + C\lambda^2\mu^2. \end{aligned}$$

It follows that there are only three independent integrals of the type $\mathcal{J}_i$, viz.

$$\begin{aligned} \mathcal{J}_1 &= L^2 + M^2 + N^2 + 2F_1, \\ \mathcal{J}_2 &= (\mu^2+\nu^2)L^2 + (\nu^2+\lambda^2)M^2 + (\lambda^2+\mu^2)N^2 + 2F_2, \\ \mathcal{J}_3 &= \mu^2\nu^2L^2 + \nu^2\lambda^2M^2 + \lambda^2\mu^2N^2 + 2F_3, \end{aligned}$$

where the F's satisfy the conditions of the last row of (25).

The result of this work may be expressed in the following terms: to determine analytically linear combinations of the first integrals of motion which form integrals of the type

$$gL^2 + hM^2 + kN^2 + 2F;$$

where g, h, k and F are functions of $(\lambda\mu\nu)$ it is necessary to carry out Eddington's analysis, for this is what we have been doing in this paragraph. It therefore appears that Jeans's is the more elegant method, since we can write down at once the expressions for the integrals of energy and momentum in rectangular Cartesian co-ordinates and then transform to the co-ordinate

system under discussion, as in § 7. Unfortunately Jeans ignores the possible existence of "invariant relations" (other than the integrals of the system) with respect to the system. When these relations exist, the distribution function derived from the integrals of motion is not the most general function. To obtain the most general form we have to take into account the "invariant relations." We now proceed to discuss this point in detail, considering only the case of motion in a plane.

10. Shiveshwarkar * has explained the deviation of the vertex direction from the direction of the galactic centre by considering a distribution function of the form

$$\exp\{-h^2(\Pi-\Pi_0)^2-m(\Pi-\Pi_0)(\Theta-\Theta_0)-k^2(\Theta-\Theta_0)^2\}.$$

Heckmann † very soon afterwards pointed out that only in special cases would this form hold. In examining this problem from the point of view of Jeans's theory we see the importance of the "invariant relations."

(*a*) *Arbitrary Central Field:* In this case there are only two integrals of motion

$$I_1=\Pi^2+\Theta^2-2\phi,$$
$$I_2=\varpi\Theta.$$

The most general form of f is

$$f(\gamma I_1+\kappa I_2{}^2+DI_2),$$

i.e.

$$f(\gamma\Pi^2+(\gamma+\kappa\varpi^2)\Theta^2+D\varpi\Theta),$$

whence (in Heckmann's notation)

$$h^2=\gamma,\qquad m=0,\qquad k^2=\gamma+\kappa\varpi^2,\qquad \Pi_0=0,\qquad \Theta_0=\frac{-\varpi D}{2(\gamma+\kappa\varpi^2)},$$

where $D\equiv$Shiveshwarkar's p.

(*b*) *Quasi-Elastic Field:* In this case

$$\phi=-E(1+\varpi^2)$$

and the integrals are

$$I_1=\Pi^2+\Theta^2-2\phi,$$
$$I_2=\varpi\Theta,$$
$$I_3=[R\cos(\theta-\omega)-\Theta\sin(\theta-\omega)]^2+2E\varpi^2\cos^2(\theta-\omega),$$
$$I_4=[R\sin(\theta-\omega)+\Theta\cos(\theta-\omega)]^2+2E\omega^2\sin^2(\theta-\omega).$$

The integrals I_3 and I_4 being the form taken by

$$u^2+2Ex^2=\text{const.},$$
$$v^2+2Ey^2=\text{const.},$$

which are the energy-integrals in the separable co-ordinates x and y. The general form of f is thus

$$f(AI_1+DI_2+\kappa I_2{}^2+2BI_3+2CI_4)$$

---

* *M.N.*, 95, 655, 1935. † *M.N.*, 96, 67, 1935.

whence

$$h^2 = A + B\cos 2(\theta-\omega) - C\cos 2(\theta-\omega) + B + C,$$
$$m = -2\sin 2(\theta-\omega)\,.\,(B-C),$$
$$k^2 = A + \kappa r^2 + B - B\cos 2(\theta-\omega) + C + C\cos 2(\theta-\omega),$$

which may be written on writing

$$\left.\begin{aligned} \gamma &= A+B+C,\\ (B-C)\cos 2\omega &= \alpha,\\ (B-C)\sin 2\omega &= \beta, \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned} h^2 &= \gamma + \alpha\cos 2\theta + \beta\sin 2\theta,\\ m &= -2\alpha\sin 2\theta + 2\beta\cos 2\theta,\\ k^2 &= \gamma + \kappa r^2 - \alpha\cos 2\theta - \beta\sin 2\theta. \end{aligned}\right\} \tag{27}$$

Equation (27) represents Heckmann's result, namely:

$$a = b = 0, \qquad \alpha \neq 0, \qquad \beta \text{ arbitrary.}$$

Writing the distribution function in the form

$$f\{h^2(\Pi-\Pi_0)^2 + m(\Pi-\Pi_0)(\Theta-\Theta_0) + k^2(\Theta-\Theta_0)^2 + \chi\},$$

where χ is a function of ϖ and θ only, we obtain on equating the co-efficients of Π and Θ

$$\left.\begin{aligned} 2h^2\Pi_0 + m\Theta_0 &= 0,\\ m\Pi_0 + 2k^2\Theta_0 &= D\Pi\,. \end{aligned}\right\} \tag{28}$$

These equations are those numbered (22) and (24) in Shiveshwarkar's paper, and his analysis for expressing $2\dfrac{(k^2-h^2)}{m}$ in terms of the A's then follows.

At this point it is desirable for us to consider the constants of integration which occur in the integrals. It is well known that when the potential is given by

$$\phi = -E(1+r^2)$$

the corresponding motion is an ellipse whose centre is the centre of force. By suitably choosing the axes of co-ordinates we may take the co-ordinates of the star at any instant to be given by

$$x = a\cos\sqrt{2E}\,.\,t, \qquad y = b\sin\sqrt{2E}\,t,$$

where a and b are the semi-axes of the ellipse.

The components of the velocity are given by

$$u = -a\sqrt{2E}\sin\sqrt{2E}\,.\,t, \qquad v = b\sqrt{2E}\cos\sqrt{2E}\,.\,t.$$

The constants appearing in I_3 and I_4 are thus seen to be

$$I_3 = u^2 + 2Ex^2 = 2Ea^2,$$
$$I_4 = v^2 + 2Ey^2 = 2Eb^2.$$

Again let h be the constant appearing in I_2, then

$$h = vx - uy.$$

When $t=0$ we have

$$x=a, \qquad y=0,$$
$$u=0, \qquad v=b\sqrt{2E}.$$

Hence

$$h=ab\sqrt{2E}. \tag{29}$$

This relation is a well-known result.*

Writing (29) in the form

$$h^2-a^2b^2\,.\,2E=0$$

we obtain the relation

$$(vx-uy)^2-2E(u^2+2Ex^2)(v^2+2Ey^2)=0. \tag{30}$$

This is an invariant relation of the system and contains no arbitrary constant. When an arbitrary constant appears the invariant relation is called a first integral and has a physical interpretation. Now in virtue of (29) the constant h is not arbitrary, it is therefore surprising to find that we have three first integrals. This is due to the fact that we are considering a very special system, namely, one in which the co-ordinates are separable. In the case of a Newtonian potential which we now examine we find that there are two integrals and an invariant relation.

(*c*) *Newtonian Potential:* For a Newtonian attraction the orbit is an ellipse for which

$$\varpi=\frac{h^2/\mu}{1+e\cos(\theta-\omega)}.$$

The integral

$$\varpi\Theta=h$$

then gives

$$\Theta=\frac{\mu}{h}(1+e\cos(\theta-\omega)). \tag{31}$$

With the aid of the integral of energy we then get

$$\Pi=\frac{\mu}{h}e\sin(\theta-\omega). \tag{32}$$

Multiplying (31) by $\cos(\theta-\omega)$ and (32) by $\sin(\theta-\omega)$ and adding we obtain

$$\Theta\cos(\theta-\omega)+\Pi\sin(\theta-\omega)=\frac{\mu}{h}\cos(\theta-\omega)+\frac{\mu e}{h}.$$

Multiplying the left-hand side by $\varpi\Theta$ and the right-hand side by h we get

$$\varpi\Theta(\Theta\cos(\theta-\omega)+\Pi\sin(\theta-\omega))=\mu\cos(\theta-\omega)+\mu e,$$

i.e.

$$\varpi\Theta(\Theta\cos(\theta-\omega)+\Pi\sin(\theta-\omega))-\mu\cos(\theta-\omega)=\mu e. \tag{33}$$

It must now be remembered that the arbitrary constants appearing in the integrals of energy and momentum are $\frac{\mu}{a}$ and h respectively. Moreover, we

* See Ramsey, *Dynamics*, p. 159, 1929.

have the well-known relation

$$h^2 = \mu a(1 - e^2).$$

It follows that the constant occurring on the right of (33) is not arbitrary. We now differentiate (33) with respect to t and verify that we have an invariant relation. Differentiating we have

$$d\varpi\Theta(\Theta\cos(\theta-\omega)+\Pi\sin(\theta-\omega))+d\theta\varpi\Theta(-\Theta\sin(\theta-\omega)+\Pi\cos(\theta-\omega))$$
$$+d\Pi\varpi\Theta\sin(\theta-\omega)+d\Theta(2\varpi\Theta\cos(\theta-\omega)+\varpi\Pi\sin(\theta-\omega))+\mu\sin(\theta-\omega)d\theta.$$

The subsidiary equations are

$$\frac{d\varpi}{\Pi}=\frac{d\Theta}{\Theta/\varpi}=\frac{d\Pi}{\frac{\Theta^2}{\varpi}-\frac{\mu}{\varpi^2}}=\frac{d\Theta}{\frac{-\Pi\Theta}{\varpi}}.$$

Substituting Π for $d\varpi$, etc. in the above expression we find that the expression vanishes identically. It therefore follows that (33) is an invariant relation, and as it does not contain an arbitrary constant it is not necessarily an integral of the system.

For a Newtonian attraction we therefore have the integrals of energy and momentum

$$I_1 = \Pi^2 + \Theta^2 - 2\phi = \frac{\mu}{a}$$

and

$$I_2 = \varpi\Theta = h,$$

together with the invariant relation

$$I_3 = \varpi\Theta(\Theta\cos(\theta-\omega)+\Pi\sin(\theta-\omega)) - \mu\cos(\theta-\omega) = \mu e.$$

We can now state our conclusion in the following terms. If we take the distribution function to be a function of the integrals of motion, the most general form of f is

$$f(\gamma I_1 + DI_2 + \kappa I_2^2). \tag{34}$$

On the other hand, taking f to be a function of the invariant relations, the most general form is

$$f(\gamma I_1 + DI_2 + \kappa I_2^2 + AI_3). \tag{35}$$

It will be noticed that (34) is only a special case of (35). Consequently we have proved that to obtain the most general form of the distribution function it is necessary to find all the invariant relations whether they are integrals or not. This can only be done by means of the method developed in § 9, and this, as we have stated, is identical with Eddington's method.

We observe that the form (35) gives in Heckmann's notation

$$\left.\begin{aligned} h^2 &= \gamma, \\ m &= A\varpi\sin(\theta-\omega) = \varpi(a\cos\theta + b\sin\theta), \\ k^2 &= \gamma + \kappa\varpi^2 + A\varpi\cos(\theta-\omega) = \gamma + \kappa\varpi^2 + \varpi(-a\sin\theta + b\cos\theta), \end{aligned}\right\} \tag{36}$$

where

$$a = -A \sin \omega,$$
$$b = A \cos \omega.$$

This is Heckmann's result, namely

$$\alpha = \beta = 0, \qquad a \neq 0, \qquad b \text{ arbitrary.}$$

Moreover, equations (28) still hold, h^2, m, k^2 now being given by (36).

11. *Summary.*—We have shown that when the distribution function is of the form

$$f = f(a^2L^2 + b^2M^2 + c^2N^2),$$

L, M, N being the velocity components along the principal directions of an orthogonal system of co-ordinates, then Jeans's method gives the most general form of the distribution function. When we turn our attention to a distribution function of the form

$$f = f(a^2L^2 + b^2M^2 + c^2N^2 + 2fMN + 2gNL + 2hLM)$$

we find that the method only gives a particular solution in general. In order to prove that we have obtained the general solution it is necessary to go through Eddington's analysis.

I should like to thank Sir Arthur Eddington and Dr. W. M. Smart for the interest they have taken in the preparation of this paper.

164 *Gilbert Road, Cambridge :*
1936 *December* 16.

THE PROPER MOTIONS OF FAINT STARS IN TWO REGIONS.

Violet F. White, M.Sc.

This paper is a continuation of four other papers,* and gives the results of measuring the photographic proper motions of faint stars in two regions whose centres are :

Region	Centre	R.A.	Dec. (1900)
		h m	° ′
VI	Orion Nebula (N.G.C. 1976)	5 31	−5 26
VII	Ring Nebula in Lyra (N.G.C. 6720)	18 50	32 56

Four plates centred on the Orion Nebula, and with the following particulars, were taken with the 24-inch Wilson reflector at the University of London Observatory:—

Plate	Date	Exposure	Observer
1	1896 Jan. 19	30 minutes	W. E. Wilson
2	1934 Jan. 19	40 ,,	C. C. L. Gregory
3	1897 Jan. 22	15 ,,	W. E. Wilson
4	1929 Dec. 29	25 ,,	C. C. L. Gregory

* *M.N.*, **92**, 519, 1932 ; **93**, 105, 1932; **93**, 647, 1933; **94**, 784, 1934.

Plates 3 and 4, having an interval of 32·9 years, were measured film to film, and Plates 1 and 2, having an interval of 38·0 years, were measured separately, the proper motions in both cases being determined from similar formulæ,

$$\mu_x = \mu_\alpha \cos\delta = M_x + a + bx + cy, \qquad \mu_y = \mu_\delta = M_y + d + ex + fy.$$

The values of the plate constants calculated by the method of least squares and by the method of Christie and Dyson were practically identical.

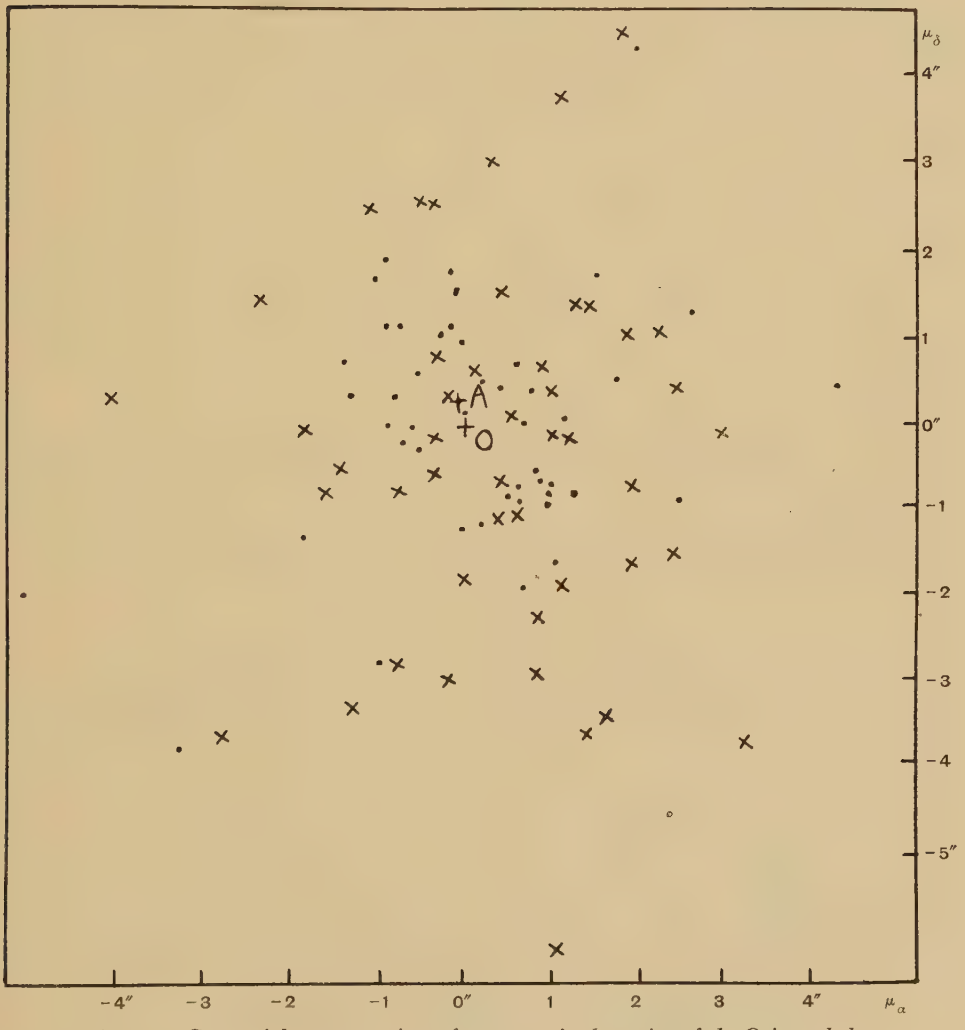

FIG. 1.—*Centennial proper motions of* 100 *stars in the region of the Orion nebula.*
Dots refer to stars with mean pg. mag. 12·5.
Crosses ,, ,, ,, 9·5.

100 stars were measured on all the plates, the faintest being of photographic magnitude 13, and the stars were divided into two groups with mean photographic magnitudes 12·5 and 9·5.

The average numerical deviation of an individual proper motion from the mean of its two determinations was 1″·2 per century in μ_α, and 0″·9 per century in μ_δ, and the probable errors of the mean centennial proper motions obtained from the two determinations were ± 0″·8 per century in μ_α, and ± 0″·6 per century in μ_δ.

The proper motions relative to the comparison stars (*i.e.* referred to origin *O*) are plotted in fig. 1.

A correction $P_x - \bar{\mu}_x$, $P_y - \bar{\mu}_y$, where P is the mean parallax, must be added to the relative proper motions to convert them to absolute proper motions. The relative proper motions and the mean parallaxes, in seconds of arc per century, for different groups of stars, are given in Table I, whence it is seen that only a very small correction to the relative proper motions is required.

TABLE I

Region	Pg. Mag	No. of Stars	H_m	P_x	$\bar{\mu}_x$	$P_x - \bar{\mu}_x$	P_y	$\bar{\mu}_y$	$P_y - \bar{\mu}_y$
			″	″	″	″	″	″	″
$VI_{12\cdot5}$	12·5	52	0·7	0·1	0·1	0·0	−0·3	0·1	−0·4
$VI_{9\cdot5}$	9·5	48	1·8	0·2	0·4	−0·2	−0·9	−0·2	−0·7
VI	10·5	100	1·1	0·1	0·2	−0·1	−0·5	0·0	−0·5
Comparison stars	12·5	37	0·7	0·1	0·0	0·1	−0·3	0·0	−0·3

The correction obtained from the comparison stars gives an origin, *A*, for the absolute proper motions, whose co-ordinates referred to *O* are −0″·1, 0″·3.

With this correction to the proper motions, the numbers of stars moving in the directions of various position angles, θ, are as given in Table II.

TABLE II

θ	Region $VI_{12\cdot5}$ No. of Stars	Region $VI_{9\cdot5}$ No. of Stars	θ	Region $VI_{12\cdot5}$ No. of Stars	Region $VI_{9\cdot5}$ No. of Stars
°			°		
0	3	0	180	1	2
10	0	1	190	0	1
20	0	4	200	1	2
30	1	0	210	0	0
40	1	0	220	2	3
50	1	2	230	1	0
60	1	0	240	2	2
70	3	3	250	1	0
80	2	1	260	1	1
90	1	1	270	2	1
100	2	2	280	1	0
110	2	3	290	1	0
120	0	1	300	1	1
130	7	1	310	1	1
140	1	1	320	2	0
150	4	4	330	1	0
160	1	4	340	1	2
170	1	1	350	2	3

From the values in this table, drift curves are drawn in fig. 2, where the positions of the apices of Drifts I and II and of the solar antapex, calculated from their positions in *M.N.*, **92**, 522, 1932, are indicated.

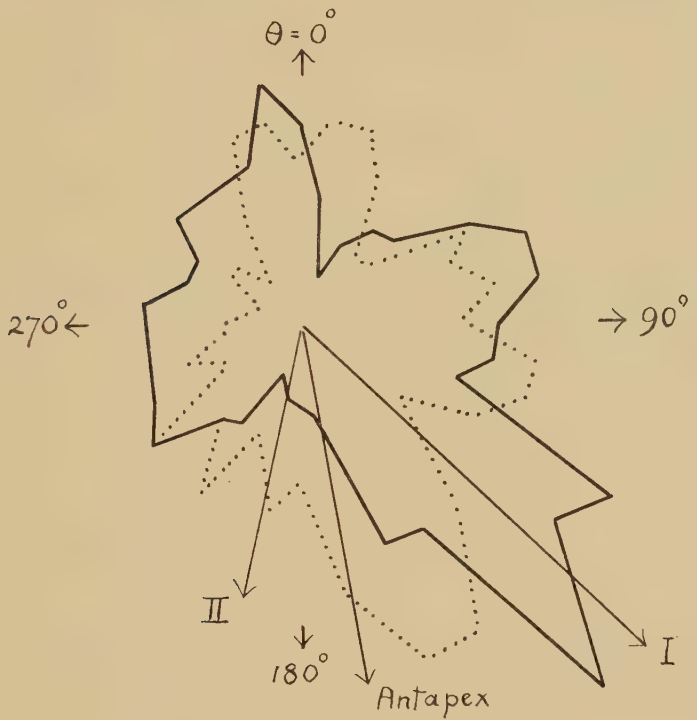

FIG. 2.

—— *refers to Region* $VI_{12\cdot5}$. *52 stars. Mean pg. mag.* 12·5.
..... *refers to Region* $VI_{9\cdot5}$. *48 stars. Mean pg. mag.* 9·5.

One star with large proper motion is omitted from the analysis; its position and centennial proper motion is :—

	(1900)			
Pg. Mag.	α	δ	μ_α	μ_δ
12	$5^h\ 30^s\cdot3$	$-5°\ 23'$	$15''\cdot6$	$-1''\cdot3$

Two plates centred on the Ring Nebula in Lyra, and with the following particulars, were measured :—

	Date	Exposure	Observer
1	1894 Aug. 4	1 hour	W. E. Wilson
2	1935 June 28	30 minutes	C. C. L. Gregory

770 stars in this region were measured, the faintest being of photographic magnitude 14·5, and the stars were divided into different magnitude groups, as in Table III. From the values in this table it is seen that no correction for the parallax of the comparison stars is necessary in order to convert the relative proper motions to absolute proper motions.

TABLE III

Region	Pg. Mag.	No. of Stars	H_m	P_x	$\bar{\mu}_x$	$P_x - \bar{\mu}_x$	P_y	$\bar{\mu}_y$	$P_y - \bar{\mu}_y$
			″	″	″	″	″	″	″
$VII_{14\cdot5}$	14·5	368	0·2	0·0	0·2	−0·2	0·0	0·4	−0·4
$VII_{13\cdot5}$	13·5	277	0·5	0·1	0·0	0·1	0·0	0·0	0·0
$VII_{12\cdot5}$	12·5	105	0·7	0·1	0·2	−0·1	0·0	−0·8	0·8
$VII_{11\cdot0}$	11·0	20	1·1	0·2	1·1	−0·9	0·0	−1·2	1·2
VII	13·8	770	0·4	0·1	0·1	0·0	0·0	0·0	0·0
Comparison stars	14·5	80	0·2	0·0	0·0	0·0	0·0	0·0	0·0

The proper motions are the mean of four measurements, the probable error of a measurement being less than 1″ per century.

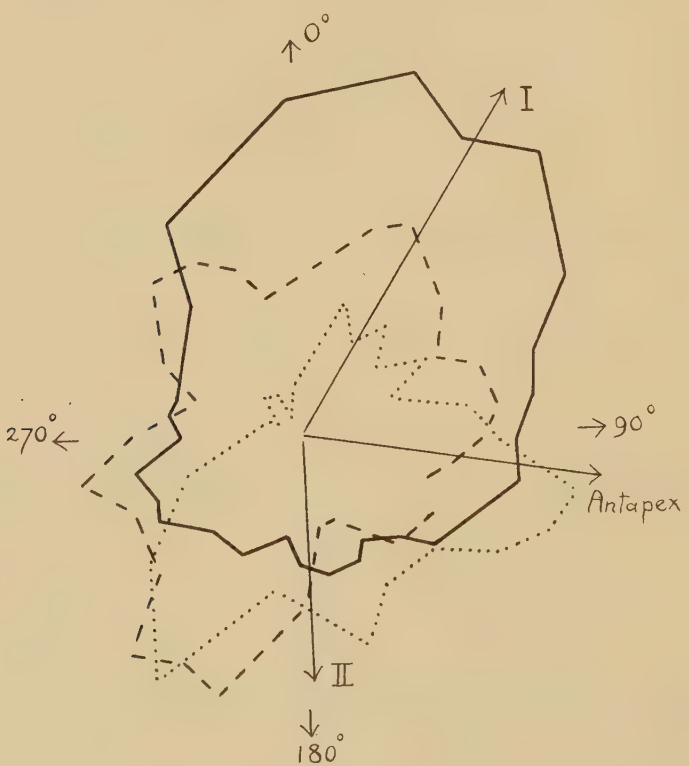

FIG. 3.

—— *refers to Region* $VII_{14\cdot5}$. 368 *stars. Mean pg. mag.* 14·5.
- - - - *refers to Region* $VII_{13\cdot5}$. 277 *stars. Mean pg. mag.* 13·5.
..... *refers to Region* $VII_{12\cdot5}$. 105 *stars. Mean pg. mag.* 12·5.

The numbers of stars of different magnitudes moving in the directions of various position angles, θ, are given in Table IV, and drift curves drawn from these values are shown in fig. 3.

TABLE IV

θ	Region $VII_{14\cdot5}$ No. of Stars	Region $VII_{13\cdot5}$ No. of Stars	Region $VII_{12\cdot5}$ No. of Stars	θ	Region $VII_{14\cdot5}$ No. of Stars	Region $VII_{13\cdot5}$ No. of Stars	Region $VII_{12\cdot5}$ No. of Stars
°				°			
0	14	4	2	180	4	6	3
10	24	9	1	190	5	12	1
20	15	9	1	200	6	8	6
30	18	10	4	210	6	15	5
40	17	11	1	220	8	9	7
50	19	5	4	230	7	11	2
60	13	9	1	240	5	4	3
70	14	5	5	250	12	9	1
80	10	10	0	260	6	9	1
90	11	10	6	270	7	10	1
100	12	3	6	280	4	3	1
110	10	7	6	290	9	7	0
120	8	6	5	300	6	4	1
130	9	6	3	310	5	8	0
140	8	6	3	320	10	11	2
150	3	6	5	330	10	8	1
160	7	2	3	340	17	7	0
170	11	3	6	350	12	7	0

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CALCULATION OF THE SCATTERING OF LIGHT FROM A SYSTEM OF SIMULTANEOUS DIFFERENTIAL EQUATIONS.

L. S. Ornstein.

(Communicated by Professor F. J. M. Stratton)

The problem of the scattering of light, which is essentially that of radiative equilibrium in grey material, has been treated with the help of an integrodifferential equation. Hopf * has indicated a mathematical treatment which enables us to calculate the radiation field by a series of functions. In discussing the derivation of the equation of the distribution of neutrons scattered in matter in velocity and space, a method occurred to me which can also be applied to the problem of light scattering and gives a process of approximation which is in principle identical with Hopf's treatment, but which has, however, the advantage of showing clear the physical sense of the process of approximation. We shall discuss the matter by considering streams of photons, as in this way the correspondence with the neutron problem is very clear—the same reasoning could be developed by the ordinary methods of optics.

Let us first treat the problem of a slab of matter which is scattering the light, the slab being bounded by two parallel infinite planes at a distance d.

* Compare, *e.g.*, E. Hopf, *Mathematical Problems of Radiative Equilibrium*, Cambridge University Press, 1934.

Let there be an incident density of q photons per unit area of surface entering at one side in a direction perpendicular to the plane. The problem is treated for the case of a steady state—the photons are assumed to have been incident since the time $-\infty$. We first shall assume that a photon has the same probability of being scattered in any direction and that the coefficient of scattering per unit time is α. Let us first consider the group of photons that suffer no scattering.

In the steady state let the number of photons at any instant in a layer between x and $x+dx$ (per unit area) be $N_0(x)$; then the condition for a steady state in the layer is

$$0 = -\alpha N_0(x) - c\frac{dN_0(x)}{dx},$$

$$N_0(x) = Ae^{-\frac{\alpha}{c}x},$$

where c is the velocity of light and A is to be determined by the condition that for $x=0$, $cN_0(0)=q$; we get therefore

$$N_0(x) = \frac{q}{c}e^{-\frac{\alpha}{c}x}.$$

Let us next consider the group of photons which have suffered only one collision. The number of photons which have co-ordinates between x and $x+dx$ and possess directions lying inside a solid angle $d\Omega$ characterised by an angle ϑ with the axis x is given by

$$N_1(x, \vartheta)d\Omega dx.$$

The condition for a steady state in the layer is

$$0 = -\alpha N_1(x, \vartheta) - c\cos\vartheta\frac{dN(x, \vartheta)}{dx} + \frac{\alpha}{4\pi}\frac{q}{c}e^{-\frac{\alpha}{c}x}.$$

The solution of this equation is

$$N_1(x, \vartheta) = Ae^{-\frac{\alpha}{c\cos\vartheta}x} + \frac{q}{4\pi c}\frac{e^{-\frac{\alpha}{c}x}}{1-\cos\vartheta}.$$

In order to find the constant A we must bear in mind that at $x=0$, $N(x, \vartheta)=0$ for $0 < \vartheta < \pi/2$, and at $x=d$, $N(x, \vartheta)=0$ for $\pi/2 < \vartheta < \pi$.

Taking this into account we find for $0 < \vartheta < \pi/2$,

$$N(x, \vartheta) = \frac{q}{4\pi c}\frac{1}{1-\cos\vartheta}\left(-e^{-\frac{\alpha}{c\cos\vartheta}x} + e^{-\frac{\alpha}{c}x}\right),$$

and for $\pi/2 < \vartheta < \pi$,

$$N^+(x, \vartheta) = \frac{q}{4\pi c}\frac{1}{1-\cos\vartheta}\left(-e^{\frac{\alpha}{c}\frac{1-\cos\vartheta}{\cos\vartheta}d - \frac{\alpha}{c\cos\vartheta}x} + e^{-\frac{\alpha}{c}x}\right).$$

The next step is the determination of $N_2(x, \vartheta)$. For this quantity we get the equation

$$0 = -\alpha N_2(x, \vartheta) - c\cos\vartheta\frac{dN_2(x, \vartheta)}{dx} + T_1,$$

where T_1 is the contribution from the scattering of the photons which have been scattered in the element. The integral consists of two parts which originate from N_1 and N_n^+; it amounts to

$$\frac{q\alpha}{2c}\left[\int_0^{\pi/2}\frac{1}{1-\cos\vartheta_1}\sin\vartheta_1 d\vartheta_1\left(-e^{-\frac{\alpha}{c\cos\vartheta_1}}+e^{-\frac{\alpha_x}{c}}\right)\right.$$
$$\left.+\int_{\frac{\pi}{2}}^{\pi}\frac{1}{1-\cos\vartheta_1}\sin\vartheta_1 d\vartheta_1\left(-e^{\frac{\alpha}{c}\frac{(1-\cos\vartheta_1)}{\cos\vartheta_1}d-\frac{\alpha}{c\cos\vartheta_1}x}+e^{-\frac{\alpha_x}{c}}\right)\right].$$

The solution of the differential equation is possible by elementary methods, and using the condition that at $x=0$ and $x=d$, N_2 is zero from 0 to $\pi/2$ and from $\pi/2$ to π respectively we can find an expression for $N_2(x, \vartheta)$.

It is clear that we can proceed in the same way and that we can find

$$N_n(x, \vartheta)$$

for values of ϑ from 0 to $\pi/2$, and

$$N_n^+(n, \vartheta)$$

for the values from $\pi/2$ to π.

A recurrence equation can easily be given; it is

$$0=-\alpha N_n(x,\vartheta)-c\cos\vartheta\frac{dN_n(x,\vartheta)}{dx}+\frac{\alpha}{2}\int_0^{\pi/2}N_{n-1}(x,\vartheta)\sin\vartheta d\vartheta$$
$$+\int_{\pi/2}^{\pi}N_{n-1}^+(x,\vartheta)\sin\vartheta d\vartheta.$$

The solution is

$$N_n(x,\vartheta)=Ae^{-\frac{\alpha}{c\cos\vartheta}x}+\frac{\alpha}{2}e^{-\frac{\alpha}{c\cos\vartheta}x}\cdot\int_0^x e^{\frac{\alpha}{c\cos\vartheta}\xi}\left[\int_0^{\pi/2}N_{n-1}(\xi,\vartheta')\sin\vartheta' d\vartheta'\right.$$
$$\left.+\int_{\pi/2}^{\pi}N_{n-1}^+(\xi,\vartheta')\sin\vartheta' d\vartheta'\right]d\xi,$$

where the constant A must be determined from the conditions that for $x=0$ $N_n(x\vartheta)=0$ for $0<\vartheta<\pi/2$ (it will therefore be zero), and that $N(d, \vartheta)=0$ for $\pi/2<\vartheta<\pi$; hence

$$A=-\frac{\alpha}{2}\int_0^d e^{\frac{\alpha}{c\cos\vartheta}\xi}\left[\int_0^{\pi/2}N_{n-1}(\xi\vartheta')\sin\vartheta' d\vartheta'+\int_{\pi/2}^{\pi}N_{n-1}^+(\xi\vartheta')\sin\vartheta' d\vartheta'\right]d\xi.$$

The formula given enables us to calculate the distribution of photons to any degree of accuracy, for we get that

$$N(\vartheta, x)=\sum_0^n N_\kappa(\vartheta, x),$$

so that the accuracy in $N(\vartheta, x)$ depends on the number of functions $N_\kappa(\vartheta, x)$ we have calculated.

We can in similar way treat the spherical problem. We shall indicate the way in which the solution can be obtained for the case in which a star is surrounded by matter scattering the light bounded by two concentric spheres

(of radii R_1 and R_2). The star is treated as a point source of light. If q photons per unit time are emitted, the number of photons per unit surface striking the inner sphere amounts to $\frac{q}{4\pi {R_1}^2}$. Now let $\rho_0(r)$ represent the density of the photons at a distance r which have suffered no collision, then we have

$$-\alpha 4\pi r^2\rho_0(r) - c4\pi\frac{d}{dr}r^2\rho_0(r) = 0,$$

$$r^2\rho_0(r) = Ae^{-\frac{\alpha}{c}r}.$$

Using the condition at the inner sphere we get for $A = \frac{q}{4\pi}e^{\frac{\alpha}{c}R_1}$ thus

$$\rho_0(r) = \frac{q}{4\pi r^2}e^{-\frac{\alpha}{c}(r-R_1)}.$$

As a next step we calculate $\rho_1(r, \vartheta)$, the density of photons in the shell between r and $r+dr$ which travel in the direction ϑ with the radius. For this quantity we get the equation

$$0 = -\alpha r^2\rho_1(r, \vartheta) - c\cos\vartheta\frac{d}{dr}r^2\rho_1(r, \vartheta) + \frac{\alpha q}{(4\pi)^2}e^{-\frac{\alpha}{c}(r-R_1)}.$$

The solution of this equation is

$$r^2\rho_1(r, \vartheta) = Ae^{-\frac{\alpha}{c\cos\vartheta}r} + \frac{q}{(4\pi)^2}\frac{e^{-\frac{\alpha}{c}(r-R_1)}}{1-\cos\vartheta}.$$

In order to find the constant A we must take into account that at the border $r=R_2$ there is no incident radiation; therefore at R_2 $\rho_1(R_2, \vartheta) = 0$ for $\pi/2 < \vartheta < \pi$; we get thus

$$r^2\rho_1^+(r, \vartheta) = \frac{q}{(4\pi)^2}\frac{1}{1-\cos\vartheta}\left(-e^{-\frac{\alpha}{c}(R_2-R_1)-\frac{\alpha}{c\cos\vartheta}(r-R_2)} + e^{-\frac{\alpha}{c}(r-R_1)}\right).$$

The solution for $0 < \vartheta < \pi/2$ can be found if we remember that the radiation scattered by the layer enters it again at the inner surface. If we consider again the angle ϑ, we find the radiation entering at this surface taking in the formula for $\rho_1^+(r, \vartheta)$ at $r=R_1$ and $\vartheta = \pi - \vartheta$. We have thus at the boundary to choose A such that

$$\rho_1(R, \vartheta) = \rho_1^+(R_1, \pi-\vartheta) = \frac{q}{(4\pi)^2}\frac{1}{1+\cos\vartheta}\left(-e^{-\frac{\alpha}{c}(R_2-R_1)+\frac{\alpha}{c\cos\vartheta}(R_1-R_2)} + 1\right).$$

In this way we get a rather complicated formula which enables us to find $\rho_1(r, \vartheta)$ for $0 < \vartheta < \pi/2$,

$$r^2\rho_1(r, \vartheta) = \frac{q}{(4\pi)^2}\left[\left(-\frac{e^{-\frac{\alpha}{c}(R_2-R_1)-\frac{\alpha}{c\cos\vartheta}R_2}}{1+\cos\vartheta} - \frac{e^{-\frac{\alpha}{c\cos\vartheta}R_1}}{1-\cos\vartheta}\right)e^{-\frac{\alpha}{c\cos\vartheta}r} + \frac{1}{1-\cos\vartheta}e^{-\frac{\alpha}{c}(r-R_1)}\right].$$

In the same way as for the first problem we can proceed to $\rho_2(r, \vartheta)$ and find a recurrence relation for $\rho_n(r, \vartheta)$. The calculations are rather tedious, but the problem can be solved in this way to any approximation.

It is possible to extend the treatment given to the case where the scattering of the photons is no longer spherical, but such that the probability of scattering into an angle $d\Omega$ is given by $\alpha f(\phi)d\Omega$, where ϕ is the angle between the direction of the incident and the scattered photon. The total scattering is given by α so that we define $f(\phi)$ such that $\int f(\phi)d\Omega = 1$.

We shall give as an example the treatment of the case of a layer limited by a plane. The equation for photons which suffered no scattering remains the same, so that $N_0(x)$ again is

$$N_0 = \frac{q}{c} e^{-\frac{\alpha}{c}x}.$$

The equation of the photons scattered once is slightly changed ; it is

$$0 = -\alpha N_1(x, \vartheta) - c \cos \vartheta \frac{dN(x, \vartheta)}{dx} + \frac{q}{c} e^{-\frac{\alpha}{c}x} \alpha f(\vartheta).$$

The solution of this equation is

$$N_1(x, \vartheta) = Ae^{-\frac{\alpha}{c \cos \vartheta}x} + \frac{1}{(1 - \cos \vartheta)} \frac{q}{c} f(\vartheta) e^{-\frac{\alpha}{c}x}.$$

The constant A can be determined again by the conditions at the limit $n = 0$ for $0 < \vartheta < \pi/2$ and $n = d$ for $\pi/2 < \vartheta < \pi$. We thus get

$$N_1^+(x, \vartheta) = \frac{1}{(1 - \cos \vartheta)} \frac{q}{c} f(\vartheta) \left(-e^{-\frac{\alpha}{c \cos \vartheta}x} + e^{-\frac{\alpha}{c}x} \right),$$

$$N_1^+(x, \vartheta) = \frac{1}{1 - \cos \vartheta} \frac{q}{c} f(\vartheta) \left(-e^{-\frac{\alpha d \cos \vartheta}{1 - \cos \vartheta} - \frac{\alpha x}{c \cos \vartheta}} + e^{-\frac{\alpha}{c}x} \right).$$

The differential equation for the second scattering is somewhat more complicated. It is

$$0 = -\alpha N_2(x, \vartheta) - c \cos \vartheta \frac{dN_2(x, \vartheta)}{dx}$$
$$+ \alpha \int_0^{\pi/2} \int_0^{2\pi} N_1(x, \vartheta_1) f(\phi) \sin \vartheta_1 d\vartheta_1 d\chi$$
$$+ \alpha \int_{\pi/2}^{\pi} \int_0^{2\pi} N_1^+(x, \vartheta_1) f(\phi) \sin \vartheta_1 d\vartheta_1 d\chi,$$

where

$$\cos \psi = \cos \vartheta_1 \cos \vartheta + \sin \vartheta_1 \sin \vartheta \cos \chi.$$

The solution is only possible if we know the form of $f(\psi)$; it is, however, then quite elementary, *e.g.* if we take Rayleigh scattering $f(\psi) = 1 + \cos^2 \psi$. By adjusting the solution to the conditions at the boundary we get again $N_2(x, \vartheta)$ from 0 to $\pi/2$ and $N_2^+(x, \vartheta)$ from $\pi/2 - \pi'$. If we introduce $N_n(x, \vartheta)$ and $N_n^+(x, \vartheta)$ for the numbers of electrons scattered n times, we get a general relation

$$0 = -aN_n(x, \vartheta) - c\cos\vartheta\frac{dN_n(x, \vartheta)}{dx} + \alpha\int_0^{\pi/2}\int_0^{2\pi} N_{n-1}(n, \vartheta')f(\psi)\sin\vartheta' d\vartheta' d\chi + \alpha\int_0^{\pi/2}\int_0^{2\pi} N^{+}_{n-1}(x, \vartheta')f(\psi)\sin\vartheta' d\vartheta' d\chi,$$

where $\cos\psi$ is given by the same relation.

It may be observed that we have not introduced in our treatment the absorption of photons. This is, however, easily possible. We only have to replace then the coefficient of loss by the sum of $\alpha_s + \alpha_a$, α_s and α_a being the probabilities of loss by scattering and by real absorption, in the first term of the differential equation. In the term which represents the production of photons of the given kind α must then be replaced by α_s, the probability of scattering. The mathematical treatment does, however, not change in any essential point.

It may be remarked that we have given our deduction in the language of photons, as it was suggested by the similar treatment of scattering of neutrons given by the author in a series of papers in the *Proceeding of the Amsterdam Academy* (1936). This, however, is unessential; it would be just the same to treat the problem on the base of beams of light scattered zero, one, two, etc. times, introducing then, instead of the photon density, the density of radiation of these different classes.

The advantage of the treatment given is that it enables us to find, to any desired degree of approximation, the solution of the problem of scattering. The calculation, however, is somewhat longer than the approximate methods used up to this moment. A great advantage is that it clearly shows the physical meaning of each step in the calculations, and gives the proof of the existence of a solution.

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A NOTE ON MOLECULAR ABSORPTION IN INTERSTELLAR SPACE.

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(Communicated by Sir Arthur Eddington)

1. For a long time the H and K lines of Ca^+ and the D line of *Na* have been the only observed stationary lines. But, recently, Merrill * has discovered four interstellar lines in the yellow and red region of the spectrum; the wave-lengths of these lines are 5780·4, 5796·9, 6283·9 and 6613·9 A. Instead of being narrow and sharp, they are somewhat widened and have rather diffuse edges.

The origin of these lines is thus far unknown. As the ultimate lines

* *P.A.S.P.*, 46, 206, 1934; *Ap. J.*, 83, 126, 1936.

of all the abundant atoms are now thoroughly known, and as none of them coincides with one of Merrill's lines, it has been suggested * that they must rather be molecular bands; the fuzzy character of the lines adds some weight to this hypothesis.

2. Actually, when trying to find a molecular identification of these lines, the types of molecular spectra to be considered are

(*a*) ultimate electronic bands;
(*b*) rotatory oscillation bands.

A systematic examination of the electronic bands likely to appear has not given us, so far, any real physical identification.

In the case of rotatory oscillation bands a striking coincidence appears. Two of Merrill's lines agree closely with the calculated wave-lengths of CO_2 bands.† The interstellar line λ 6613·9 ($\nu = 15115{\cdot}5$ cm.$^{-1}$) coincides with the origin of the band ($\nu_1 + 4\nu_2 + 5\nu_3$) at $\nu = 15115$; another stationary line λ 6283·9 ($\nu = 15909{\cdot}3$ cm.$^{-1}$) falls near the calculated CO_2 band at 15917 cm.$^{-1}$ ($7\nu_3$). The second coincidence is not so good, but it is admissible owing to the fact that the wave-numbers of the CO_2 bands are only calculated (not yet observed in the laboratory), and also owing to the diffuse character of the stationary lines.

This is of course a purely tentative identification. It will be possible to find an observational criterion. As these bands correspond to rather high vibrational quantum numbers, they must be weaker than the "Venus bands" at $\lambda\lambda$ 7820 and 7883, and the latter bands ought to be strong in the spectra of all sufficiently distant stars. Dr. Merrill, to whom this coincidence has been announced, will be on the look-out for further evidence, and hopes to undertake a search for the "Venus bands" in the future.‡

That the other calculated bands of CO_2 have not been observed as interstellar lines may be due to lower transition probabilities, unfortunate blendings,§ or technical difficulties.

The numerical coincidences given here may be purely accidental; future observations will decide it. They are merely cited as an illustration of the actual interest of the consideration of the interstellar molecules.

3. An immediate objection to a molecular identification of the lines observed by Merrill is that these observed lines, though fuzzy, are not broad enough for molecular origin. For instance, the CO_2 bands when observed in the laboratory average about 100 cm.$^{-1}$ in breadth throughout the spectrum.‖

This objection disappears when we assume a very low temperature for the Boltzmann distribution of the molecules in the different rotational levels of the lowest electronic state. Owing to the selection rule concerning the rotational quantum number $\mathcal{J}$ ($\mathcal{J}' - \mathcal{J}'' = \pm 1$ or sometimes 0), only a few

* H. N. Russell, *M.N.*, **95**, 635, 1935.
† Adel and Dennison, *Physical Review*, **43**, 723, 1933.
‡ Private communication.
§ *I.e.* the band at $\nu = 15234$ (λ 6562·4) falls on $H\alpha$.
‖ Kindly communicated by Dr. Adel.

individual lines of the absorption bands will appear, and actually only those corresponding to small quantum numbers $\mathcal{J}$; in other words, only a few lines near the band origins will appear.

4. Let us consider first the case of a molecule which has a pure rotational spectrum, *i.e.* which has an electric dipole in the normal electronic state. We know that the speeds of the electrons in interstellar space correspond to a temperature of the order 10,000°, and the translatory velocities of the molecules will be brought up to equilibrium with the electrons. But in this case, owing to the strong selective absorption between the different rotational levels, we must consider for each rotational absorption line λ an equivalent temperature T_λ, exactly as in the treatment of atomic excitations * ; this equivalent temperature T_λ is such that the actual energy-density for wave-length λ is equal to that of equilibrium radiation at temperature T_λ.

We shall use the following formula for the molecular energy of rotation :

$$E=\frac{h^2}{8\pi^2 I}\cdot\mathcal{J}(\mathcal{J}+1) \qquad (\mathcal{J}=0,\ 1,\ 2,\ \ldots), \tag{1}$$

I being the moment of inertia and $\mathcal{J}$ the rotational quantum number.

We know from the experimental investigation of the infra red spectra of pure rotation that these appear in the region of wave-lengths greater than 10 μ. For small moments of inertia (or great energies E, following formula (1)), the spectra lie between 10 and 100 μ; for higher values of I—*e.g.* of the same order as that of CO_2—the first rotational lines have still greater wave-lengths, which may be calculated by the usual formula,

$$\nu=\frac{h}{4\pi^2 cI}(\mathcal{J}+1) \qquad (\mathcal{J}=0,\ 1,\ 2,\ \ldots). \tag{2}$$

For I of the order of $70\cdot10^{-40}$ g. cm.^{2} (value for CO_2), this would give for the first ten lines values extending from about $\lambda=10{,}000$ μ to $\lambda=1000$ μ. It seems therefore useful to estimate the equivalent temperatures T_λ for $\lambda=10$, 100 and 1000 μ.

For the determinations of T_λ we have followed Eddington's procedure. For $\lambda=1000$ μ, T_λ is already smaller than $T=3°{\cdot}2$ K.† For 10 and 100 μ we obtain

$$T_{\lambda=10\mu}=43°\ K\ ;$$
$$T_{\lambda=100\mu}=4°{\cdot}8\ K.$$

Thus for the lowest value of $\mathcal{J}$ the extreme value of T_λ to be applied—even for the smallest moments of inertia I—is about 5° K. For the very high values of $\mathcal{J}$, T_λ could be as high as 43° K, at least for very small moments of inertia.

Table I gives the population for $T_\lambda=10$, 5 and $3°{\cdot}2$ K, corresponding to $I=70\cdot10^{-40}$ g. cm.^{2} and computed according to Boltzmann's formula,

$$N_J=(2\mathcal{J}+1)e^{-\frac{E_J}{kT}}. \tag{3}$$

* A. S. Eddington, *Internal Constitution of Stars*, 2nd ed., p. 371.

† This is the value of T obtained by A. S. Eddington by applying the formula $E=aT^4$ to the total radiation of the stars ($7{\cdot}67\cdot10^{-13}$ ergs/cm.^{3}).

TABLE I

Populations of the rotational levels of a dipolar molecule, for $T_\lambda = 10°$, $5°$ *and* $3°{\cdot}2$ *K, assuming* $I = 70{\cdot}10^{-40}$ *g. cm.*^{2}.

j	Population for $T_\lambda = 10°$ K	Population for $T_\lambda = 5°$	Population for $T_\lambda = 3°{\cdot}2$
0	1	1	1
1	2·67	2·37	2·07
2	3·55	2·55	1·75
3	3·57	1·82	0·84
4	2·97	0·99	0·27
5	2·01	0·36	0·055
6	1·17	0·12	0·009
7	0·61	0·027	...
8	0·31	0·005	...

Evidently these populations do not exactly correspond to the intensities of the individual lines of the absorption bands; for these intensities we should also have to consider the transition probabilities (Hönl-London's formulæ). But this would not modify substantially our conclusions.

Table I shows that in our assumptions only the four or five first lines of the P, Q and R branches will be really observable; this means that the breadth of the bands will be only 6 or 7 cm.$^{-1}$, or approximately 2 A. near 6000 A.; the edges must be diffuse. This seems to agree perfectly with Merrill's observations.

5. Let us now consider a molecule—such as CO_2—which has no electric dipole in the lowest electronic state. It seems at first that the collisions will bring the rotations up to a distribution corresponding to a temperature of the order 10,000°. But this is not necessarily true, as the molecule may still have a "quadrupolar transition probability" for the pure rotational spectrum, which may be appreciable; thus, owing to the very low density of interstellar matter, the molecule may still be able to absorb and emit these low-frequency radiations faster than the exchanges by collision processes. There are still other reasons which indicate that the consideration of a low temperature for the distribution of the rotational quantum numbers is appropriate.

Consequently results similar to those of Table I would also be expected here.

6. When once the identifications have been definitely made the determination of the profile of the interstellar bands will give a possibility of direct measure of the temperatures corresponding to the actual Boltzmann distribution of the rotational levels.

It is quite clear that our estimations of T_λ are rather rough; * on the

* For instance, in our calculations of T_λ we have not considered any star of temperature lower than 3000°.

other hand, the strict application of the equivalent temperatures T_λ to the Boltzmann distributions correspond only to ideal cases.

7. So far no distinction has been made between the di- and polyatomic molecules. We have only considered the general molecular case, with an application to CO_2. Actually, the combinations of atoms into diatomic molecules in a radiation field require collision processes and are only possible for compounds which have adequate potential curves $U(r)$. For polyatomic molecules the problem is still more complicated, at least when it does not reduce to the diatomic case as a first approximation (one atom being combined with a very strongly bound diatomic group).

The problem of the dissociation equilibrium of molecules in a radiation field has some similarity with the problem of the ionization equilibrium of atoms in a radiation field, though there is an important difference. In the atomic case all the photons of energy $h\nu$ greater than the ionization potential of an atom A are able to ionize A. For the molecules only discrete regions of wave-lengths can give rise to dissociation.

This shows the interest of a first formal attempt at a theoretical treatment of this problem. We hope to investigate this question in the near future.

My thanks are due to Professor Sir Arthur Eddington and to Dr. S. Chandrasekhar for valuable suggestions.

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ON THE NUMBER OF INCOMPRESSIBLE FLUID SPHERES OF GIVEN RADIUS.

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(Communicated by the Secretaries)

Introduction

1. In a paper "On the Equilibrium of an Incompressible Sphere" * Prof. N. R. Sen has calculated numerically the sizes of spheres of constant proper densities, which are in equilibrium under gravitational forces. The pressure depends on a certain parameter ξ which decreases from its central value ξ_0 steadily outwards, and attains its minimum value unity on the free surface of the sphere. The conclusions arrived at are, that corresponding to every (proper) density there is a sphere of maximum size of finite mass, and also a (different) sphere of maximum gravitational mass of finite size. Besides, for any given radius and density, there are two spheres of which one is the classical sphere and another a non-classical sphere whose central pressure is greater than that of the classical sphere. The numerical calculations from which these results were derived were carried on from $\xi_0 = 1{\cdot}01$ to $\xi_0 = 10$. The sizes of the spheres increased from $\xi_0 = 1{\cdot}01$

* Sen, *M.N.*, **94**, 550, 1934.

steadily till $\xi_0 = 1{\cdot}90$, when the maximum radius (in suitable measure) $R = 0{\cdot}996$ was reached. For larger values of ξ_0 the size decreased again and a steady decrease was maintained as far as the calculations were made, namely, up to $\xi_0 = 10$. It was then conjectured that as ξ would increase, the size would decrease continually to zero, as the equations can be shown to possess solutions such that $r \to 0$ as $\xi \to \infty$.

In this paper some results of calculation with higher values of ξ_0 are incorporated. It has been found advisable to restrict the investigation to the determination of the number of complete spheres for a given radius. Detailed calculation shows that below a certain critical radius $R = 0{\cdot}727$ (in suitable measure) there is only one sphere, namely, the classical sphere, and the maximum size of the sphere previously found ($R = 0{\cdot}996 \sim 1$) is correct. Between these two values there are two regions within one of which there are *two* spheres, and within the other *three* spheres. It is conjectured that this is the maximum number, and the very large number of numerical calculations performed supports this view.

An Upper Limit to the Size of the Sphere

2. It was proved theoretically by Sen that R must be less than $\sqrt{3} = 1{\cdot}73$, and his numerical calculations reached a maximum $R_{\max} = 0{\cdot}996 \sim 1$. Since all the principal results have been obtained by numerical calculations from non-linear differential equations whose behaviour appears to be very complicated, it is worth while to establish an upper limit to the size of the sphere by an analytical method.

The following proof * brings down the limit from 1·73 to 1·25.

The differential equations are

$$\frac{1}{r^2}\frac{dw}{dr} = \tfrac{1}{4}(3\xi^2 + 1), \tag{1}$$

$$\frac{1}{r}\frac{dz}{dr} = -\xi^3, \tag{2}$$

with

$$w/r + z/\xi = 1. \tag{3}$$

The quantities r, w, z, ξ are all pure numbers, r being the representative of the distance, w of the gravitational mass, ξ of the pressure and z of the light velocity in the co-ordinate system used. The free surface is $\xi = 1$, ξ decreasing continuously from its central value ξ_0 to unity; w increases from zero value at $r = 0$ to a finite value when $\xi = 1$ and $r = R$; in fact, as $r \to 0$, $w = o(r^3)$. z decreases from its central value outwards. Physical necessity requires the quantities to be all positive, and w to be restricted by $w < r$.

Writing (3) as

$$\xi = \frac{z}{1 - \dfrac{w}{r}}, \tag{3'}$$

* This proof is due to Professor Sen.

differentiating with respect to r, and then substituting for dw/dr, dz/dr from (1) and (2) and simplifying by (3′) we have

$$\left(1-\frac{w}{r}\right)\left(-\frac{d\xi}{dr}\right)=\frac{w\xi}{r^2}+\tfrac{1}{4}r\xi^3-\frac{r\xi}{4}. \tag{4}$$

Equation (1) gives

$$w=\tfrac{1}{4}\int_0^r(3\xi^2+1)r^2dr>\tfrac{1}{12}(3\xi^2+1)r^3 \tag{1′}$$

since ξ is monotone decreasing from $r=0$ outwards, whence we have

$$\frac{w}{r}>\tfrac{1}{12}(3\xi^2+1)r^2, \tag{a}$$

a much more rough approximation being

$$\frac{w}{r}>\tfrac{1}{3}r^2. \tag{b}$$

Using this on the left of (4),

$$(1-\tfrac{1}{3}r^2)\left(-\frac{d\xi}{dr}\right)>r\left[\xi\frac{w}{r^3}+\tfrac{1}{4}\xi(\xi^2-1)\right],$$

and further, using (*a*) on the right for w/r^3, we have

$$\left(-\frac{d\xi}{dr}\right)>\tfrac{1}{6}\cdot\frac{r\xi(3\xi^2-1)}{1-\frac{1}{3}r^2}.* \tag{c}$$

Again from (1′) we have by partial integration

$$w=\tfrac{1}{12}r^3(3\xi^2+1)+\tfrac{1}{2}\int_0^r r^3\xi\left(-\frac{d\xi}{dr}\right)dr,$$

whence using (*c*) we get the inequality

$$w>\tfrac{1}{12}r^3(3\xi^2+1)+\tfrac{1}{12}\xi^2(3\xi^2-1)\int_0^r\frac{r^4}{1-\frac{1}{3}r^2}dr.$$

This on integration gives

$$\frac{w}{r}>\tfrac{1}{12}r^2(4\xi^2-3\xi^4+1)-\tfrac{3}{4}\xi^2(3\xi^2-1)+\frac{3\sqrt{3}}{8}\,\frac{\xi^2(3\xi^2-1)}{r}\log\frac{1+\frac{1}{\sqrt{3}}r}{1-\frac{1}{\sqrt{3}}r}. \tag{1″}$$

For our purpose it will be convenient to replace ξ by its minimum value 1. That this is allowable can be easily seen from the equation preceding (1″). Thus

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* By integration from this we can get an upper limit $R_{\max}<1{\cdot}29$, but a still smaller value is obtained by the analysis given above.

$$\frac{w}{r} > \tfrac{1}{6}r^2 - \tfrac{3}{2} + \frac{3\sqrt{3}}{4}\frac{1}{r}\log\frac{1+\frac{1}{\sqrt{3}}r}{1-\frac{1}{\sqrt{3}}r}. \qquad (d)$$

Let us replace w/r on the left of (4) by (d) and w/r on the right of (4) by (1′), we then have an inequality which can be integrated thus,

$$-6\int_{\xi_0}^{1}\frac{d\xi}{\xi(3\xi^2-1)} > \int_0^R \frac{rdr}{\frac{5}{2} - \frac{1}{6}r^2 - \frac{3\sqrt{3}}{4}\cdot\frac{1}{r}\cdot\log\frac{1+\frac{1}{\sqrt{3}}r}{1-\frac{1}{\sqrt{3}}r}},$$

$\xi_0(>1)$ being the central value of ξ and R the value of r on the surface where $\xi = 1$. The left-hand side is a monotone increasing function of ξ_0 and tends to the limit 1·216 as $\xi_0 >> 1$. The right-hand side is also a monotone increasing function of R and, using a graphical method, we find that the maximum R consistent with the above inequality is given by

$$R_{\max} = 1{\cdot}25.$$

Thus the upper limit to the radius of the sphere is brought down to 1·25.

Equation (d) is interesting, as it gives a lower limit to the value of w (gravitational mass of the sphere) on the surface for a given value of the radius R. We write this as

$$\frac{w_s}{R} > \tfrac{1}{6}R^2 - \tfrac{3}{2} + \frac{3\sqrt{3}}{4}\cdot\frac{1}{R}\cdot\log\frac{1+\frac{1}{\sqrt{3}}R}{1-\frac{1}{\sqrt{3}}R}. \qquad (d')$$

We shall make use of this result in the approximate integration performed later on. Again $w \geqslant r$ is prohibited. From (1″) it is evident that corresponding to a given value of r, say r_0, we can find an upper limit to ξ (since $w \leqslant r_0$) for r_0. Hence for a given "distance" r from the centre there is an upper limit to the value of the pressure (ξ_l) at that point for spheres having radii $> r$. Thus in the (ξ, r) plane there is a (ξ_l, r) curve which no (ξ, r) curve should cross from below.

3. As our object is to find the number of complete spheres of *a given radius* we take a sphere of a certain radius R, start the integration from the surface where ($\xi = 1$), and carry it backward towards $r \to 0$. It is evident that an arbitrary value of w on the surface, which we shall denote by w_s, will give one configuration. The corresponding solution will not in general give a complete sphere, as w in such cases in general will vanish for $r > 0$. These w_s's give shell solutions. Our problem then is to find out the number of the initial values of w_s which will give complete spheres, for which, when the integration is carried backward, w and r will vanish together. The choice of initial w_s is fortunately limited by the relation (d') on the lower side, and by

$w_s < R$ on the upper side. The method adopted in this paper is to perform the integrations for a definite R for a series of values of w_s between these two limits backwards, and to find out the cases where complete spheres are obtained. The continuous nature of the (w, r) solution usually makes the

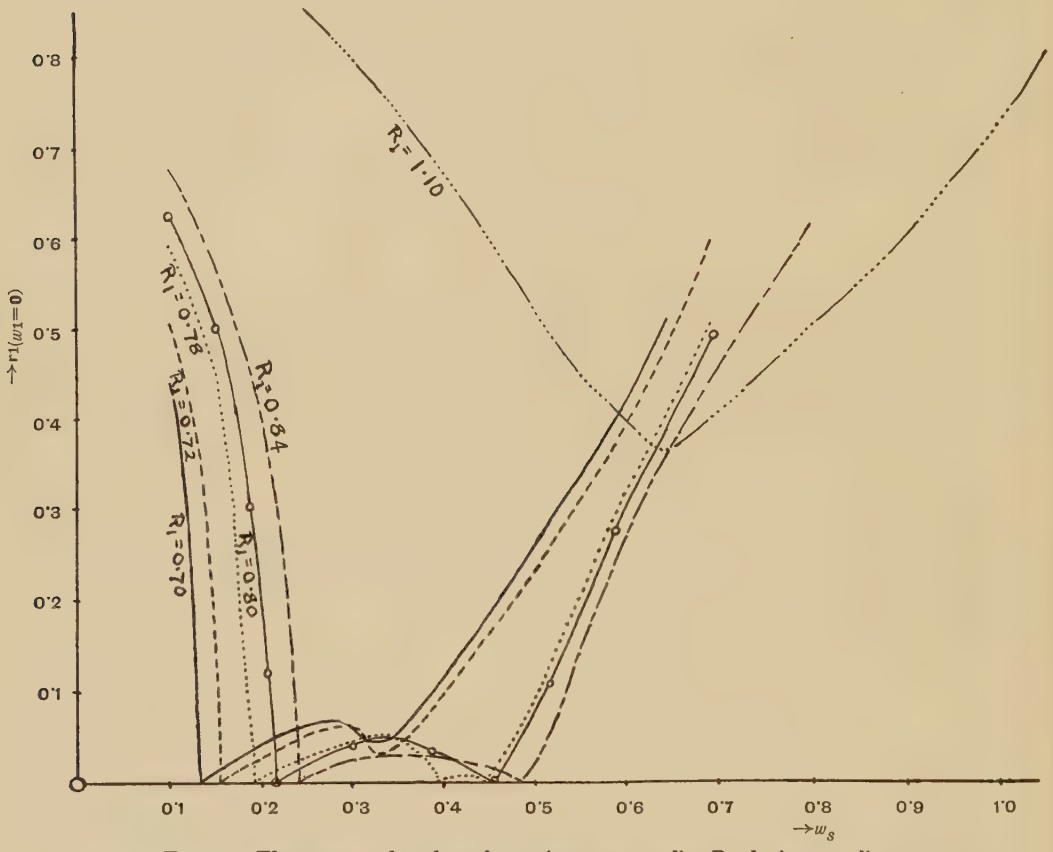

FIG. 1.—*These curves show how for a given outer radius R_1 the inner radius $r_{1(w=0)}$ changes for varying total mass w_s of the shell. $r_{1(w=0)} = 0$ means a complete sphere. The total number of spheres of a definite radius R_1 is given by the number of times the corresponding curve meets the w_s axis. Thus there is one sphere of radius $R_1 = 0{\cdot}70$, there are two spheres of $R_1 = 0{\cdot}84$, and three spheres of $R_1 = 0{\cdot}78$, etc. There is no sphere of $R_1 = 1{\cdot}10$.*

detection of a complete sphere from the integration chart easy. Thus for a given R a series of integrations with values of w lying between two limits must be completed before we can answer the question how many spheres of that particular radius R exist. This involves long work, which has been considerably simplified by the following rough approximations. Instead of a rigorous integration, the nature of the solution was guessed from the following approximate calculation :—

$$\left.\begin{aligned} \Delta w &= \tfrac{1}{4}(3\xi^2 + 1)r^2\Delta r, \\ \Delta z &= -r\xi^3\Delta r, \\ w/r + z/\xi &= 1. \end{aligned}\right\}$$

Starting from a definite w_s (usually the lowest limit given by (d')) at a certain $r = R$ where $\xi = 1$, the successive steps were calculated backwards from the above difference relations till w for a certain $r = r_{w=0}$ vanished. Next a slightly higher value of w_s was taken and the whole process was repeated. It was found that $r_{w=0}$ varies in a continuous manner, which represents a characteristic of the (w, r) integral curves. The values of $r_{w=0}$ were not accurate, but our main object, namely, to find out when w and r vanish together, was achieved by examining how the values of $r_{w=0}$ behave. Whenever these values appeared to decrease and approach zero, accurate integrations were performed in the neighbourhood, and the existence or non-existence of a complete sphere was verified. It often happened that $r_{w=0}$ attained a minimum without this minimum value actually vanishing. These cases do not represent complete spheres, and have to be carefully eliminated. It has been found that the approximately and rigorously integrated values of (w, r) show the best agreement in the neighbourhood of a complete sphere. In general, the ξ values show wide divergence (in spite of w, r values agreeing moderately). The reason probably lies in the fact that the ξ function shows discontinuity sometimes attaining infinite values.

The results of the numerical calculation can be briefly stated as follows :—

It is known * that increasing central pressure gives a corresponding continuously increasing radius, R (from $R = 0$ to $R = \cdot 996$), for the spheres. These spheres are called "classical" by Sen. Further increase of central pressure causes the radius to decrease. These may be called non-classical spheres of type A. Our calculation shows that this reduction proceeds until $R = \cdot 727$, with one series of non-classical spheres. Further increase of central pressure causes the radius to increase again, up to $R = \cdot 786$. Thus 20 integrations performed for $R = \cdot 72$ gave only one sphere, the classical one for that radius. There are therefore no non-classical spheres of very small radius. The types of spheres are as follows :—

(1) Classical with radii between 0 and ·996,
(2) Non-classical A ,, ·996 and ·727,
(3) ,, B ,, ·727 and ·786.

Fig. 1 shows the number of complete spheres of different types for certain radii for which calculations were made. They helped to indicate the limits of the different series of spheres clearly.

In conclusion, I express my thanks to Prof. N. R. Sen for guidance and help.

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* Sen, *M.N.*, 94, 550, 1934.

MICROMETRICAL MEASURES OF DOUBLE STARS.

(28TH SERIES.)

W. Milburn.

The micrometrical measures are given as in previous lists, all being made with the 24-inch (Calver) reflector.

Name	B.D.	1900 R.A.	1900 Decl.	P.A.	Dist.	Mags.	Nights	Date 1936 +	
	°	h m	° ′	°	″				
Es 153	+40·42	0 11·2	+40 44	245·3	2·56	9·5, 10·0	2	·034	
Es 1486	+40·77	19·9	40 48	349·6	6·05	...	2	·794	
Es 1938	+36·52	20·0	36 51	10·7	6·73	...	2	·794	
Es 1603	+40·80	20·9	40 40	328·8	2·16	...	2	·828	BC
				55·6	...	...	1	·810	AB
Es 1939	+38·54	23·3	38 32	280·5	6·36	...	2	·851	
Es 1941	+37·92	29·4	37 44	350·1	2·59	...	2	·034	
M 3	...	41·7	38 28	21·0	7·41	...	3	·905	
M 4	+38·114	43·0	38 51	100·8	6·23	...	2	·876	AB
				168·8	17·78	...	2	·876	AC
Es 1605	+40·174	45·3	41 3	275·4	5·68	...	1	·975	
Es 1489	+40·181	46·9	40 59	26·5	4·82	...	1	·975	
M 971	...	52·4	39 14	19·3	4·14	9·5, 10·5	2	·905	
M 972	...	52·5	39 12	288·4	4·94	10·5, 11·0	2	·905	
Es 1947	+37·245	1 10·7	37 35	44·9	5·90	...	3	·423	
Es 1494	+40·323	29·5	40 52	11·6	5·19	...	2	·034	
Es 1609	+39·392	38·4	40 8	199·8	2·31	...	2	·805	
M 12	+38·418	2 3·0	39 7	256·4	6·39	...	2	·034	
Es 1610	(+40·446)	3·4	40 27	291·1	2·19	11·0, 11·2	2	·306	BC
				124·4	...	...	2	·423	AB
Es 164	+40·475	11·9	40 24	333·3	8·43	...	2	·851	
M 13	+39·600	34·6	39 48	137·5	7·75	...	2	·101	
Es 1510	+40·637	53·0	40 46	323·4	4·20	...	1	·975	
Σ 360	+36·650	3 5·8	36 50	131·3	2·31	...	2	·501	
Σ 369	+39·743	10·6	40 7	33·7	3·33	...	2	·101	
Es 2147	+37·764	18·8	38 4	48·9	2·53	...	3	·932	
AG 70	+36·735	35·8	36 57	29·8	6·79	...	2	·501	AB
				348·0	...	...	1	·887	AC
M 16	+38·802	39·0	38 50	317·7	8·37	...	2	·034	
Σ 500	+39·945	4 4·8	40 0	81·6	3·95	...	2	·114	
Pettit. 8	+37·902	8·7	38 1	97·6	2·75	...	2	·114	
M 18	...	13·1	40 14	59·0	2·78	...	1	·975	
Es 2086	...	14·0	38 22	207·2	2·99	...	2	·034	
M 19	+39·987	18·8	39 12	218·6	2·47	...	3	·929	
Es 2149	+38·911	27·9	38 11	221·0	5·31	...	2	·034	
M 974	...	5 4·3	37 2	295·0	7·32	...	3	·099	AB
				352·9	20 ±	...	2	·092	AC
Es 60	+40·1261	14·5	40 44	274·2	6·08	...	2	·172	
Es 61	+40·1263	15·1	40 41	359·9	2·43	...	3	·440	
M 975	+37·1602	6 45·5	37 11	37·6	2·10	...	1	·033	AB
				83·4	25 ±	C = 10·7	1	·115	AC
Es 2102	+37·1797	7 47·8	37 2	270·5	6·92	...	1	·137	
Es 179	...	50·3	37 59	92·2	4·63	...	2	·114	

Name	B.D.	1900 R.A.	1900 Decl.	P.A.	Dist.	Mags.	Nights	Date 1936+
	°	h m	° ′	°	″			
Es 2103	+37·1811	52·0	37 16	193·0	2·59	...	2	·126
Es 2012	+38·1915	8 23·7	38 35	91·3	5·62	...	1	·112
Hu 1192	+39·3546	18 46·8	39 55	50·3	2·44	...	2	·649
Es 1659	+40·3605	19 5·8	40 48	255·1	6·65	...	3	·645
Es 1668	+39·3856	35·2	40 6	136·8	2·50	...	2	·802
Es 1673	+40·3947	52·1	40 33	194·8	3·79	...	3	·645
Es 1971	+39·4037	20 3·3	39 21	290·7	3·33	...	2	·772
M 25	+40·4298	41·1	41 9	208·9	5·37	...	1	·652
Es 1989	+39·4439	21 3·0	39 57	173·4	4·63	...	2	·649
Es 2059	+38·4351	3·1	38 15	267·3	7·10	9·5, 10·0	2	·810
h 1703	+39·4703	50·0	39 24	83·5	8·80	...	2	·719
Es 1700	...	22 41·0	39 33	241·6	4·55	...	3	·780
Es 1702	+40·4948	51·8	40 22	265·0	2·78	...	2	·787
Es 1998	+37·4731	52·6	38 0	90·9	3·92	...	2	·917
Es 2000	+39·5003	23 1·0	39 33	68·9	6·79	...	2	·917
Es 267	...	25·8	38 57	187·5	2·25	...	2	·805
Es 2005	+36·5129	49·3	36 27	158·8	5·16	...	3	·791

New Double Stars

Milburn No.	B.D.	1900 R.A.	1900 Decl.	P.A.	Dist.	Mags.	Nights	Date 1936+	
	°	h m	° ′	°	″				
998	...	1 46·1	+38 23	335·7	3·99	11·8, 11·9	3	·930	
999	...	46·8	39 46	46·6	2·56	10·5, 11·2	2	·876	
1000	...	2 8·1	39 30	90·4	5·31	12·0, 13·0	2	·961	
1001	...	33·6	39 41	26·5	5·40	10·2, 11·2	2	·891	AB
				250·2	20 ±	C = 11·2	1	·895	AC
1002	...	53·2	38 25	92·3	5·86	10·0, 11·5	2	·034	
1003	...	4 43·9	39 16	274·8	5·74	12·0, 13·0	3	·040	
1004	...	44·0	39 16	123·6	5·93	10·0, 11·5	3	·015	
1005	...	55·1	35 53	223·8	10·44	10·0, 10·5	1	·964	
1006	...	5 29·5	36 20	90·6	2·37	10·0, 11·0	2	·969	
1007	...	30·5	37 0	74·3	3·36	10·5, 11·0	2	·092	
1008	...	30·6	36 59	39·1	3·40	11·0, 13·0	1	·137	
1009	...	6 4·3	35 44	123·0	3·70	11·5, 13·5	2	·126	
1010	...	4·4	35 44	104·3	4·10	11·0, 11·5	2	·074	
1011	...	10·5	37 2	204·9	8·34	10·0, 10·0	1	·964	
1012	...	19 52·0	40 33	8·3	4·14	11·0, 11·2	2	·642	
1013	...	20 12·7	39 49	162·0	2·84	10·5, 10·8	2	·656	
1014	...	16·2	39 39	238·9	4·69	10·7, 11·5	1	·840	
1015	...	16·3	39 41	87·2	3·15	12·0, 12·1	2	·764	
1016	...	16·7	39 40	18·3	2·25	9·7, 9·8	2	·839	
1017	...	22·4	38 32	22·3	2·81	11·0, 11·2	2	·649	
1018	...	21 3·3	38 17	247·8	5·19	10·0, 11·0	2	·723	
1019	+38·4354	3·4	38 17	248·4	5·00	9·0, 10·5	2	·723	
1020	...	13·0	38 30	169·5	3·73	11·0, 12·0	2	·642	
1021	...	22·7	39 31	84·1	6·48	10·0, 12·0	2	·653	
1022	...	54·9	39 32	243·2	7·30	10·0, 12·0	5	·873	
1023	...	55·3	39 30	237·8	5·25	10·5, 10·6	2	·873	
1024	...	55·9	39 32	210·0	7·22	10·1, 10·5	2	·872	
1025	...	22 45·6	39 56	170·8	5·19	11·0, 12·0	3	·948	

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OCCULTATIONS OF STARS BY THE MOON OBSERVED AT BIRMINGHAM UNIVERSITY DURING THE YEAR 1935.

James Young.

This report is a continuation of those of previous years.

Table I gives the usual particulars of the observations. The U.T.'s are uncorrected for lag. Except for No. 153 the phenomena were all observed under good conditions.

Table II gives the reductions. A correction of $0^h.00253$ has been added to the U.T.'s for the purpose of taking the Moon's right ascension, declination and parallax from the *Nautical Almanac*, as suggested by Prof. E. W. Brown.

Table III gives the positions of the stars not in the *Nautical Almanac* for 1935, together with the authorities. Two positions have been taken from the *Nautical Almanac* for 1936.

TABLE I

No.	Star	Mag.	Phase	Date	U.T.
				d	h m s
136	B.D. +23° 1648	9·6	D	Feb. 14	19 50 41·0
137	B.D. +23° 1647	8·4	D	Feb. 14	19 54 37·5
138	B.D. +25° 804	7·8	D	Mar. 11	21 52 47·0
139	B.D. +25° 807	7·9	D	Mar. 11	22 38 50·6
140	B.D. +25° 1100	7·0	D	Mar. 12	18 28 51·2
141	B.D. +25° 1108	8·0	D	Mar. 12	19 11 23·7
142	B.D. +25° 1121	8·8	D	Mar. 12	20 15 30·3
143	B.D. +25° 1125	8·0	D	Mar. 12	20 36 55·1
144	B.D. +25° 1133	8·1	D	Mar. 12	21 14 34·9
145	B.D. +25° 1131	7·6	D	Mar. 12	21 22 17·9
146	B.D. +18° 1939	8·1	D	Apr. 11	21 07 25·3
147	B.D. +18° 1947	8·3	D	Apr. 11	22 10 32·4
148	ρ Aqu	5·4	R	Jul. 18	22 38 24·2
149	λ Sag	2·9	D	Sept. 07	19 27 29·2
150	20 H′ Ari	6·4	R	Sept. 15	21 28 15·8
151	22 B Pis	6·5	D	Nov. 06	18 36 57·3
152	9 Pis	6·4	D	Nov. 06	21 27 48·4
153	8 Gem	6·1	R	Nov. 13	20 02 49·8
154	9 Gem	6·3	R	Nov. 13	20 43 44·1
155	18 Aqu	5·5	D	Dec. 01	18 37 28·4
156	B.D. −6° 5972	7·5	D	Dec. 02	21 56 49·2
157	B.D. −6° 5974	8·1	D	Dec. 02	22 10 26·6
158	47 Ari	5·8	D	Dec. 08	00 37 44·6

TABLE II

No.	$\sigma'-\sigma$	χ	ρ	$\chi-\rho$	No.	$\sigma'-\sigma$	χ	ρ	$\chi-\rho$
	″	° ′	° ′	° ′		″	° ′	° ′	° ′
136	−2·2	109 48	101 52	+ 07 56	140	−0·7	131 02	94 22	+ 36 40
137	−2·4	145 53	101 53	+ 44 00	141	−1·1	30 33	94 33	− 64 00
138	−1·7	85 57	89 06	− 03 09	142	−2·0	89 45	94 49	− 05 04
139	−2·5	67 35	89 18	− 21 43	143	−2·3	98 14	94 54	+ 03 20

No.	$\sigma'-\sigma$	χ	ρ	$\chi-\rho$	No.	$\sigma'-\sigma$	χ	ρ	$\chi-\rho$
	″	° ′	° ′	° ′		″	° ′	° ′	° ′
144	−1·4	74 05	95 03	− 20 58	152	−1·4	358 46	64 50	− 66 04
145	−0·5	32 48	95 05	− 62 17	153	+2·6	355 14	96 00	−100 46
146	−2·6	164 10	108 28	+ 55 42	154	+0·8	277 16	96 10	−178 54
147	−0·4	39 46	108 38	− 68 52	155	−1·0	77 44	67 57	+ 09 47
148	+2·2	217 55	65 03	+152 52	156	−1·9	14 35	65 34	− 50 59
149	−2·0	45 37	82 45	− 37 08	157	−1·7	11 51	65 33	− 53 42
150	+1·8	229 07	71 19	+157 48	158	−2·7	147 36	76 15	+ 71 21
151	−1·5	46 35	64 48	− 18 13					

TABLE III

No.	$\alpha_{1935\cdot0}$	$\Delta\alpha$	$\delta_{1935\cdot0}$	$\Delta\delta$	Authority
	h m s	s	° ′ ″	″	
136	7 11 15·88	+2·79	+23 21 06·3	+ 0·1	Y
137	7 10 57·29	+2·79	+23 13 27·7	+ 0·1	Y
138	5 10 30·03	+1·77	+25 50 36·7	+ 8·0	Y, G
139	5 11 59·18	+1·78	+25 53 07·9	+ 7·9	Y, G
140	6 00 10·91	+2·09	+25 26 53·6	+ 5·7	*N.A.*, 1936
141	6 01 03·82	+2·10	+25 48 21·4	+ 5·7	Y, G
142	6 03 22·87	+2·11	+25 31 40·5	+ 5·5	Y, G
143	6 03 58·21	+2·11	+25 27 57·7	+ 5·5	Y, G
144	6 05 01·13	+2·12	+25 31 26·5	+ 5·4	Y, G
145	6 04 46·35	+2·12	+25 39 25·8	+ 5·5	Y, G
146	8 22 12·81	+2·29	+18 02 48·1	− 4·3	Ab
147	8 24 11·14	+2·30	+18 17 03·6	− 4·3	Ab, C
156	22 17 59·78	+3·72	− 6 34 14·5	+21·9	*N.A.*, 1936
157	22 18 22·49	+3·72	− 6 30 32·5	+22·0	Ab, Al

Ab = Abbadia 1900.
Al = Albany 1910.
C = Cincinnati 1900.
G = Greenwich 1910.
Y = Yale photographic.

The University,
Edgbaston, Birmingham :
1936 *December* 13.

NOTE ON STELLAR IONIZATION AND ELECTRIC FIELDS.

Rupert Wildt.

(Communicated by Professor E. A. Milne)

Despite the tendency of electrons and ions to separate in a gravitational field according to their masses, ionized stellar matter is electrically neutral, practically. The distribution of electric field-strength, which prevents the separation of charges in an isothermal star, is known as the Pannekoek-Rosseland field. Milne * gave a more detailed analysis of this field in a one-component system, and Fowler † discussed qualitatively a two-component system. The influence of stellar rotation and a superimposed magnetic field of solar character was evaluated by Cowling.‡ A further

* E. A. Milne, *Proceedings of the Cambridge Philosophical Society*, **22**, 493, 1924.
† R. H. Fowler, *Statistical Mechanics*, p. 365, 1929.
‡ T. G. Cowling, *M.N.*, **90**, 140, 1929.

modification, though scarcely of great numerical significance, is required by the presence of a radial temperature gradient.

The treatment of the ionization equilibrium under a temperature gradient, as recently given by the writer,* is incomplete, and will be repeated here in a more adequate way. Nevertheless, acquaintance with that paper seems desirable for the understanding of the following terse exposition. If $-e$ denotes the charge of the electron, E the electric field-strength and $Z_{ai}=i$, the "valency" of the ith ionization state of the atomic species a, in diffusion equilibrium the distribution of the partial pressures is described by the equations

$$\frac{dp_{ao}}{dx}=-\frac{gm_{ao}}{kT}p_{ao}+\phi_{ao}p_{ao} \tag{1a}$$

for the atoms,

$$\frac{dp_{ai}}{dx}=-\frac{gm_{ai}}{kT}p_{ai}-Z_{ai}eEp_{ai}+\phi_{ai}p_{ai} \tag{1b}$$

for the ions,

$$\frac{dp_e}{dx}=-\frac{gm_e}{kT}p_e+eEp_e+\phi_e p_e \tag{1c}$$

for the electrons. Here the functions ϕ_{ai} serve to satisfy the condition of ionization equilibrium (local thermodynamic equilibrium) throughout the star:

$$\phi_{ai}+\phi_e-\phi_{a(i-1)}=\frac{dlnK_{ai}}{dx}. \tag{2}$$

The conservation of hydrostatic equilibrium requires

$$\phi_e p_e+\sum_1^a\sum_0^i\phi_{ai}p_{ai}=0. \tag{3}$$

The equations (2) and (3) are not sufficient to determine the ϕ's; rather certain kinetic considerations are to be taken into account. Relative to any level in the atmosphere, there is, under a temperature gradient, an excessive recombination of ions and electrons at the higher levels and an excessive ionization at the lower ones. Each constituent has a radial component of motion different from zero, and the steady process of diffusion transports reaction energy. Thus, for each chemical component (electrons and various species of atomic nuclei) an equation of continuity has to be imposed, and these make the problem definite. With the U's denoting the respective diffusion velocities, the equations of continuity of the plane problem take the form

$$\sum_0^i U_{ai}p_{ai}=0, \tag{4}$$

$$U_e p_e+\sum_1^a\sum_0^i(Z_a^*-Z_{ai})U_{ai}p_{ai}=0, \tag{5}$$

where Z_a^* is the atomic number (nuclear charge) of the ath element. The difference $(Z_a^*-Z_{ai})$ gives the number of bound electrons per nucleus ai,

* R. Wildt, *Astrophysical Journal*, 83, 202, 1936.

and $Z_{ao}=0$ refers to the neutral atoms. The sum of the equations (4) and (5) is

$$U_e p_e + \sum_1^a \sum_0^i (Z_a^* - Z_{ai} + 1) U_{ai} p_{ai} = 0. \tag{6}$$

Then a comparison of the coefficients of (3) and (6) reveals the proportionalities

$$U_e \sim \phi_e \quad \text{and} \quad (Z_a^* - Z_{ai} + 1) U_{ai} \sim \phi_{ai},$$

from which (4) and (5) can be transformed to

$$\sum_0^i \frac{1}{(Z_a^* - Z_{ai} + 1)} \phi_{ai} p_{ai} = 0, \tag{7}$$

$$\phi_e p_e + \sum_1^a \sum_0^i \frac{Z_a^* - Z_{ai}}{Z_a^* - Z_{ai} + 1} \phi_{ai} p_{ai} = 0. \tag{8}$$

Now the equations (2), (7) and (8) are sufficient to eliminate the ϕ's. The difference from the treatment given in the paper quoted above is that the presence of the higher ionization states has been taken care of. This complicates the solution in a way overlooked before, namely, the solution for any one-component system does not approach asymptotically that for hydrogen, if the partial pressures of all ionization states higher than the first one vanish. Therefore, the equations (27) of the former paper * do not give the ϕ's for any binary mixture of singly ionized elements, but they are valid only for a mixture of the two hydrogen isotopes, which implies $\frac{dlnK_1}{dx} = \frac{dlnK_2}{dx}$. To illustrate this point further, the ϕ's for pure hydrogen, pure helium and a mixture of hydrogen and helium are given in the following, the isotopic constitution of these elements being neglected.

Hydrogen :

$$p_{11} \cdot p_e = p_{10} \cdot K_{11} \quad (p_{11} : H^+, \qquad p_{10} : H).$$

$$\phi_{10} = -\frac{2 p_{11} p_e}{p_{10}(p_{11} + p_e) + 2 p_{11} p_e} \cdot \frac{dlnK_{11}}{dx}, \tag{9a}$$

---

* Unfortunately some terms of these equations are wrong, as noticed recently by the writer. The complete correct formulæ are :

$$\phi_1 = \frac{1}{D}\left[-2p_3(p_5(p_2+2p_4)+p_2p_4)\frac{dlnK_1}{dx} + 2p_2p_3p_4\frac{dlnK_2}{dx}\right], \tag{27a}$$

$$\phi_2 = \frac{1}{D}\left[-2p_4(p_5(p_1+2p_3)+p_1p_3)\frac{dlnK_2}{dx} + 2p_1p_3p_4\frac{dlnK_1}{dx}\right], \tag{27b}$$

$$\phi_3 = \frac{1}{D}\left[+p_1(p_5(p_2+2p_4)+p_2p_4)\frac{dlnK_1}{dx} - p_1p_2p_4\frac{dlnK_2}{dx}\right], \tag{27c}$$

$$\phi_4 = \frac{1}{D}\left[+p_2(p_5(p_1+2p_3)+p_1p_3)\frac{dlnK_2}{dx} - p_1p_2p_3\frac{dlnK_1}{dx}\right], \tag{27d}$$

$$\phi_5 = \frac{1}{D}\left[+p_1p_3(p_2+2p_4)\frac{dlnK_1}{dx} + p_2p_4(p_1+2p_3)\frac{dlnK_2}{dx}\right], \tag{27e}$$

$$D = p_5(p_1+2p_3)(p_2+2p_4) + p_1p_3(p_2+2p_4) + p_2p_4(p_1+2p_3).$$

$$\phi_{11} = +\frac{p_{10}p_e}{p_{10}(p_{11}+p_e)+2p_{11}p_e}\cdot\frac{dlnK_{11}}{dx}, \tag{9b}$$

$$\phi_e = +\frac{p_{10}p_{11}}{p_{10}(p_{11}+p_e)+2p_{11}p_e}\cdot\frac{dlnK_{11}}{dx}. \tag{9c}$$

Helium :

$$p_{22}\cdot p_e = p_{21}\cdot K_{22},\quad p_{21}\cdot p_e = p_{20}\cdot K_{21}\quad (p_{22}: He^{++},\quad p_{21}: He^{+},\quad p_{20}: He).$$

$$\phi_{20} = \frac{1}{D}\left[-3(p_{21}(p_{22}+p_e)+2p_{22}p_e)\frac{dlnK_{21}}{dx}+3p_{22}(p_{21}-2p_e)\frac{dlnK_{22}}{dx}\right], \tag{10a}$$

$$\phi_{21} = \frac{1}{D}\left[+2p_{20}(2p_{22}+p_e)\frac{dlnK_{21}}{dx}-2p_{22}(2p_{20}+3p_e)\frac{dlnK_{22}}{dx}\right], \tag{10b}$$

$$\phi_{22} = \frac{1}{D}\left[-p_{20}(p_{21}-2p_e)\frac{dlnK_{21}}{dx}+(p_{20}(p_{21}+2p_e)+3p_{21}p_e)\frac{dlnK_{22}}{dx}\right], \tag{10c}$$

$$\phi_e = \frac{1}{D}\left[+p_{20}(p_{21}+4p_{22})\frac{dlnK_{21}}{dx}+p_{22}(4p_{20}+3p_{21})\frac{dlnK_{22}}{dx}\right], \tag{10d}$$

$$D = p_{20}(p_{21}+8p_{22}+2p_e)+3(p_{21}(p_{22}+p_e)+2p_{22}p_e).$$

Mixture of hydrogen and helium :

$$D\cdot\phi_{10} = -12p_{11}[p_e(p_{21}+2p_{22})+p_{21}p_{22}]\frac{dlnK_{11}}{dx}-4p_{11}p_{20}(p_{21}+8p_{22}+2p_e)\frac{dlnK_{11}}{dx}$$
$$+4p_{11}p_{20}(p_{21}+4p_{22})\frac{dlnK_{21}}{dx}+4p_{11}p_{22}(4p_{20}+3p_{21})\frac{dlnK_{22}}{dx}, \tag{11a}$$

$$D\cdot\phi_{11} = +6p_{10}[p_e(p_{21}+2p_{22})+p_{21}p_{22}]\frac{dlnK_{11}}{dx}+2p_{10}p_{20}(p_{21}+8p_{22}+2p_e)\frac{dlnK_{11}}{dx}$$
$$-2p_{10}p_{20}(p_{21}+4p_{22})\frac{dlnK_{21}}{dx}-2p_{10}p_{22}(4p_{20}+3p_{21})\frac{dlnK_{22}}{dx}, \tag{11b}$$

$$D\cdot\phi_{20} = +6p_{10}p_{11}(p_{21}+4p_{22})\frac{dlnK_{11}}{dx}-6[(p_{10}+2p_{11})(p_e(p_{21}+2p_{22})+p_{21}p_{22})$$
$$+p_{10}p_{11}(p_{21}+2p_{22})]\frac{dlnK_{21}}{dx}$$
$$-12p_{22}[p_e(p_{10}+2p_{11})+p_{10}p_{11}]\frac{dlnK_{22}}{dx}+6p_{21}p_{22}(p_{10}+2p_{11})\frac{dlnK_{22}}{dx}, \tag{11c}$$

$$D\cdot\phi_{21} = -4p_{10}p_{11}(p_{20}-3p_{22})\frac{dlnK_{11}}{dx}+2p_{20}(p_{10}+2p_{11})(4p_{22}+2p_e)\frac{dlnK_{21}}{dx}$$
$$+4p_{10}p_{11}p_{20}\frac{dlnK_{21}}{dx}$$
$$-12p_{22}[p_e(p_{10}+2p_{11})+p_{10}p_{11}]\frac{dlnK_{22}}{dx}-8p_{20}p_{22}(p_{10}+2p_{11})\frac{dlnK_{22}}{dx}, \tag{11d}$$

$$D \cdot \phi_{22} = -2p_{10}p_{11}(4p_{20}+3p_{21})\frac{dlnK_{11}}{dx} - 2p_{20}(p_{10}+2p_{11})(p_{21}-2p_e)\frac{dlnK_{21}}{dx} + 4p_{10}p_{11}p_{20}\frac{dlnK_{21}}{dx} + 6p_{21}[p_e(p_{10}+2p_{11})+p_{10}p_{11}]\frac{dlnK_{22}}{dx} + 2p_{20}[2p_{10}p_{11}+(p_{10}+2p_{11})(p_{21}+2p_e)]\frac{dlnK_{22}}{dx}, \quad (11e)$$

$$D \cdot \phi_e = +2p_{10}p_{11}(2p_{20}+3p_{21}+6p_{22})\frac{dlnK_{11}}{dx} + 2p_{20}(p_{10}+2p_{11})(p_{21}+4p_{22})\frac{dlnK_{21}}{dx} + 2p_{22}(4p_{20}+3p_{21})(p_{10}+2p_{11})\frac{dlnK_{22}}{dx}, \quad (11f)$$

$$D = 6[p_e(p_{10}+2p_{11})(p_{21}+2p_{22})+p_{10}p_{11}(p_{21}+2p_{22})+p_{21}p_{22}(p_{10}+2p_{11})] + 4p_{10}p_{11}p_{20}+2p_{20}(p_{10}+2p_{11})(p_{21}+8p_{22}+2p_e).$$

If the partial pressure p_{20} of neutral helium is put zero in the equations (10), the remaining terms reduce exactly to the form of the equations (9) $\phi_{21}=\phi_{10}$ and $\phi_{22}=\phi_{11}$, p_{21} being identified with p_{10} and p_{22} with p_{11}. This shows clearly the hydrogen-like structure of ionized helium. In the same way, the equations (11) turn to the corrected equations (27)—compare footnote on page 227—if p_{20} vanishes.

By multiplying the equations (4) each with the corresponding Z_a^* and subtracting all from (5), it becomes obvious that the local charge density is conserved, while the process of diffusion goes on :

$$U_e p_e - \sum_1^a \sum_0^i Z_{ai} U_{ai} p_{ai} = 0. \quad (12)$$

A transformation from the U's to the ϕ's gives

$$\phi_e p_e - \sum_1^a \sum_0^i \frac{Z_{ai}}{Z_a^* - Z_{ai} + 1}\phi_{ai} p_{ai} = 0. \quad (13)$$

This equation should be compared with another one,

$$\phi_e p_e - \sum_1^a \sum_0^i Z_{ai}\phi_{ai} p_{ai} = 0, \quad (14)$$

which expresses the condition that the additional pressure gradients ϕp set up by the temperature gradient do not disturb the Pannekoek-Rosseland field. Evidently (13) and (14) are mutually consistent only for $Z_a^* = 1$, *i.e.* in a star composed solely of the hydrogen isotopes.

The condition (14) is easily derived from the Poisson equation of the plane problem (N the negative charge density, n_e and n_{ai} the numbers of the electrons and ions per cubic centimetre),

$$\frac{1}{4\pi e}\cdot\frac{dE}{dx} = N = n_e - \sum_1^a \sum_0^i Z_{ai} n_{ai}, \quad (15)$$

which gives by differentiation:

$$\frac{dN}{dx} = -\frac{g}{kT}\left[m_e n_e - \sum_1^a \sum_0^i Z_{ai} m_{ai} n_{ai}\right] + \frac{eE}{kT}\left[n_e + \sum_1^a \sum_0^i Z_{ai}^2 n_{ai}\right] + \left[\phi_e n_e - \sum_1^a \sum_0^i Z_{ai}\phi_{ai} n_{ai}\right] - \frac{dlnT}{dx}\left[n_e - \sum_1^a \sum_0^i Z_{ai} n_{ai}\right]. \quad (16)$$

By substitution from (15) and taking account of $m_{ai} = m_{ao} - Z_{ai} m_e$, this may be changed to

$$\frac{dN}{dx} = -\frac{g}{kT}\left[m_e\left(N + \sum_1^a \sum_0^i (Z_{ai}+1) Z_{ai} n_{ai}\right) - \sum_1^a m_{ao} \sum_0^i Z_{ai} n_{ai}\right] + \frac{eE}{kT}\left[N + \sum_1^a \sum_0^i (Z_{ai}+1) Z_{ai} n_{ai}\right] + \left[\phi_e n_e - \sum_1^a \sum_0^i Z_{ai}\phi_{ai} n_{ai}\right] - N\frac{dlnT}{dx}. \quad (17)$$

Since the star must be practically neutral electrically, an approximation for the electric field is gained from (17) by putting N and dN/dx equal to zero, $\phi_e n_e$ being substituted from (3):

$$E = -\frac{g}{e}\left[\frac{\sum_1^a m_{ao} \sum_0^i Z_{ai} n_{ai}}{\sum_1^a \sum_0^i (Z_{ai}+1) Z_{ai} n_{ai}} - m_e\right] - \frac{kT}{e}\left[\frac{\sum_1^a \sum_0^i (Z_{ai}+1)\phi_{ai} n_{ai}}{\sum_1^a \sum_0^i (Z_{ai}+1) Z_{ai} n_{ai}}\right]. \quad (18)$$

The first term is the Pannekoek-Rosseland field in a form slightly different from that which Eddington * gave. The relative importance of the second term is determined by the weighted mean of $kT\phi_{ai}$, each of which quantities may be written as sum of terms proportional to the various $\frac{dlnK_{ai}}{dx}$, thus

$$kT\phi_{ai} = \sum f(\ldots, n_{ai}, \ldots) kT \frac{dlnK_{ai}}{dx} = \sum f(\ldots, n_{ai}, \ldots) \frac{I_{ai}}{T} \cdot \frac{dT}{dx}. \quad (19)$$

The second of these equations follows from

$$\frac{dlnK_{ai}}{dT} = +\frac{I_{ai}}{kT^2} + \ldots, \quad (20)$$

higher terms of the sum of states, where I_{ai} is the ionization energy ($1.59 \cdot 10^{-12}$ ergs per volt).

On introducing numerical values corresponding to the stellar surface into the equations (9), it becomes obvious that for hydrogen the ϕ's compete with the isothermal terms gm/kT and cause remarkable deviations from the ordinary distribution of partial pressures (barometric-height formula). With heavy atoms the influence of the isothermal terms, proportional to the atomic mass, prevails. Since the linear temperature gradient is proportional to the gravity acceleration, this is true for both giants and dwarfs. In the stellar interior the highly ionized matter has ionization potentials running up to hundreds and thousands of volts, which leads to a proportional increase

* A. S. Eddington, *Internal Constitution of the Stars*, p. 274, 1926.

of the ϕ's. These may effectively counteract the isothermal terms as regards the distribution of the partial pressures of the various ions. But for the electric field-strength, the process of averaging the ϕ's in the second term of (18) probably causes only a rather small correction of the Pannekoek-Rosseland field. A detailed examination of this point is not feasible without involved numerical calculations. But the qualitative validity of the conclusion just drawn becomes evident from the fact that the individual ϕ's have both positive and negative signs and, therefore, cancel each other at least partially.

Institute for Advanced Study,
Princeton, New Jersey:
1936 *November.*

THE DIMENSIONS OF NOVA HERCULIS.

Arthur Beer, Ph.D.

1. We consider a star of which the brightness, the spectral type and the radial velocity (given by the displacements of the absorption lines) change systematically in such a way that with time the type becomes later, the brightness increases, while the velocity of approach is decreasing. This case is realised in several new stars (notably Nova Gem 1912, Nova Aql 1918, Nova Her 1934) before their principal maximum.

2. Taking the time-interval between t_1 and t_2 we can use the observed values of the temperature T_1 and of the magnitude m_1 at t_1, and of T_2 and m_2 at t_2. Assuming as an approximation that Planck's law is valid, we can apply the well-known formula * which connects absolute visual magnitude M with temperature T and radius ξ of the stars (expressed in units of Sun radii) :—

$$M = \frac{29500}{T} - 5 \log \xi - 0{\cdot}08. \qquad (1)$$

Subtracting the equation obtained for the time t_2 from the corresponding one for t_1, and replacing $(M_1 - M_2)$ by the difference of the observed apparent magnitudes $(m_1 - m_2)$, we obtain

$$\log \frac{\xi_2}{\xi_1} = \frac{5900}{T_2} - \frac{5900}{T_1} + \frac{m_1 - m_2}{5}. \qquad (2)$$

3. In the case of Nova Pictoris 1925 a similar relation was used by H. Spencer Jones † in a calculation of the dimensions of this star. In a recent paper D. B. McLaughlin ‡ applied it to six novæ. To illustrate his

* Russell-Dugan-Stewart, *Astronomy*, p. 732, 1927.
† H. Spencer Jones, *Cape Ann.*, **10**, pt. 9, p. 163, 1931.
‡ D. B. McLaughlin, *A.J.*, **45**, 145, 1936.

method we have to take as ξ_1 the radius of the star at the time τ_1 of its first radial velocity observation, and as ξ_2 the radius at the time τ_2 of its light maximum. The velocity curve furnishes the relation

$$\xi_2' = \xi_1' + \int_{\tau_1}^{\tau_2} V dt, \tag{3}$$

where V is the velocity of approach. McLaughlin assumes that equation (3) is supplementing for equation (2), if the latter is applied for the moments τ_1 and τ_2. In other words, the postulate is introduced, $\xi_1' = \xi_1$, $\xi_2' = \xi_2$. Then

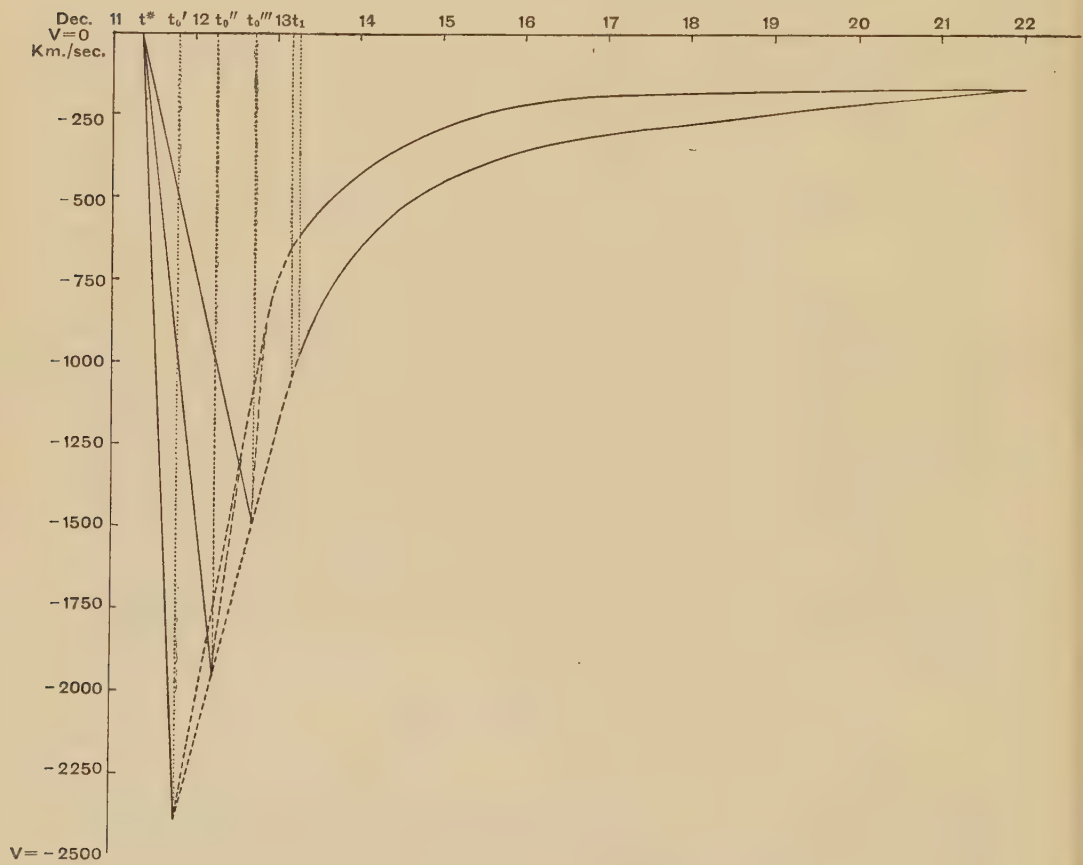

FIG. 1.—*Velocity curves of Nova Herculis* 1934 *between outburst and light maximum.* (*Lower curve : Hydrogen. Upper curve : Fe*II, *etc.*)

the combination of equations (2) and (3) gives ξ_1 and ξ_2, and the second value used for ξ in equation (1) yields the absolute magnitudes of the stars at maximum. A comparison of the resulting parallaxes with other determinations seemed to give some confidence in the general applicability of the method.

Three assumptions underlie the above procedure : (1) That the velocity curve represents the real rate of the expansion of the star ; (2) that the temperature can be deduced from the observed absorption spectrum ; and

(3) that the equation (1) which is valid for "normal stars" can also be extended to novæ.

4. We examine here the special case of Nova Herculis 1934. The numerous observations between discovery and maximum available for this nova * enable us to follow the velocity changes much more closely than in any previous nova.

Discussing an investigation of H. N. Russell † on the deceleration of expanding shells around novæ, F. J. M. Stratton ‡ reproduces velocity curves for the "Absorption I" of Nova Herculis from the time of discovery, 1934 December 13·27, up to December 27. The interesting result is that he obtains two separate curves, one for hydrogen with greater negative velocities than the other for ionised iron, etc. For the present calculation we shall have to deal, therefore, separately with the V_H and $V_{Fe\,II,\,etc.}$ values; these are represented in fig. 1.

In this figure, some details of which are to be explained below, §§ 11, 12, V is used up to 1934 December 21·77 only. This is the last date on which it is safe to assume undisturbed velocities of absorption I alone, because afterwards an absorption II could be measured also; if these absorptions are ascribed to different "shells," the two may affect one another after this date. It seems especially noteworthy that the two curves for absorption I, which were continually approaching each other, came together and showed a nearly common velocity of -170 km./sec. for hydrogen, iron, titanium, etc., just at the date when absorption II appeared.

5. Table I gives in successive columns:

t_n = dates for which the following values are taken from smooth curves drawn through the individual observations.

V_H = radial velocity of hydrogen.

$V_{Fe\,II,\,etc.}$ = radial velocity of ionised iron, etc.

m = observed magnitudes § given by a smoothed light-curve.‖

T_G = temperatures derived from the Greenwich photometric measures of the continuous spectrum.¶

T_R' = the values T_G reduced to the temperature system of Russell-Dugan-Stewart.**

T = the temperatures in this system as inferred from the observed spectral type. For this purpose the Cambridge plates were inspected closely with α Cyg, A2, and ε Aur, F2, as comparison

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* F. J. M. Stratton, *Annals, Solar Physics Observatory, Cambridge*, 4, pt. 4, 1936; W. S. Adams, W. H. Christie, A. H. Joy, R. F. Sanford, O. C. Wilson, *P.A.S.P.*, 47, 206, 1935.

† H. N. Russell, *P.A.S.P.*, 48, 29, 1936.

‡ F. J. M. Stratton, *The Observatory*, 59, 327, 1936.

§ A. Beer, *M.N.*, 95, 538, 1935, gives the curve through the individual values, which are affected by the rapid fluctuations of the star, occurring especially in this first phase.

‖ H. Krumpholz, *Sitzungsberichte Akademie d. Wiss. Wien*, IIa, 144, 512, 1935.

¶ W. H. M. Greaves and E. Martin, *M.N.*, 96, 428, 1936.

** *Handbuch d. Astrophysik*, VII, 459, 1936.

spectra. The general agreement between T_R' and T for the last four dates gives one confidence in using T throughout the whole range, *i.e.* also for the earlier days, when T_R' is not available.

TABLE I

t_n	V_H	$V_{FeII, etc.}$	m	T_G	T_R'	T
1934 Dec. 13·27	−1050	−625	3·85	...	...	15,000
14·0	− 650	−430	3·70	...	...	12,150
15·0	− 450	−280	3·48	...	...	11,000
16·0	− 360	−212	3·18	...	...	10,200
17·0	− 312	−190	2·92	...	...	9,500
18·0	− 275	−180	2·86	...	...	8,850
19·0	− 240	−173	2·68	11,100	8,000	8,250
20·0	− 212	−172	2·40	10,600	7,700	7,750
21·0	− 190	−171	2·20	10,200	7,450	7,250
21·77	− 175	−170	1·82	9,850	7,250	6,950

6. It seems of interest to supplement D. B. McLaughlin's calculations (*loc. cit.*), applying his method to the earlier dates now available. His method, consisting in the combination of equations (2) and (3), yields (denoting the right side of (2) by a and the integral in (3) by b),

$$\xi_1 = \frac{b}{10^a - 1} \quad \text{and} \quad \xi_2 = \xi_1 + b. \tag{4}$$

The velocity curve is here to be integrated between $\tau_1 =$ December 13·27, and the time of maximum $\tau_2 =$ December 22·19. For this last date we have $T = 6750$ and $m = 1{\cdot}32$.

Table II gives the results separately for hydrogen and *Fe*II, etc., using McLaughlin's hypotheses A and B. The figures in column B are those

TABLE II

	H		*Fe*II, etc.	
	A	B	A	B
ξ_1	61·0	43·6	41·4	29·6
ξ_2	590·7	421·9	401·7	286·9
M	− 13·3	− 12·6	− 12·5	− 11·7
p	0·00012	0·00017	0·00018	0·00025

given directly by the equations and the observed velocity curve, while those in A are the values obtained after the velocities have been multiplied by the factor 1·4 to allow for integration over the disc * of the effect on the

* A. S. Eddington, *Internal Constitution of the Stars*, p. 185, 1926; J. A. Carroll, *M.N.*, **88**, 551, 1928.

displacement of the absorption lines caused by uniform expansion * and limb darkening.

The absolute magnitude at maximum appears now much too high, the parallax differing from the generally accepted value † by a factor of more than 10, and hence doubts as to the consistency of the procedure arise.

7. To examine the situation more closely we divide the time-interval into one-day periods, and compute the limiting radii for each interval independently. We have then ξ_n corresponding to the time t_n, and ξ_{n+1} to t_{n+1}; again hydrogen and *Fe*II, etc. are treated separately.

TABLE III

t_n, t_{n+1}	*H*			*Fe*II, etc.		
	$b_{n,\,n+1}$	ξ_n	ξ_{n+1}	$b_{n,\,n+1}$*	ξ_n*	ξ_{n+1}*
Dec. 13·27–14·0	73·1	223·5	296·6	47·5	145·2	192·7
14·0 –15·0	66·2	274·7	340·9	42·7	177·2	219·9
15·0 –16·0	49·4	187·1	236·5	29·8	112·9	142·7
16·0 –17·0	41·6	170·5	212·1	24·7	101·2	125·9
17·0 –18·0	36·3	255·6	291·9	23·0	161·9	184·9
18·0 –19·0	32·1	150·8	182·9	22·0	103·3	125·3
19·0 –20·0	28·1	106·4	134·5	21·4	81·1	102·5
20·0 –21·0	24·9	104·7	129·6	21·1	88·6	109·7
21·0 –21·77	17·5	60·1	77·6	16·3	56·0	72·3

It is very interesting to compare the values of ξ_{n+1} (or ξ_{n+1}*) of each line of Table III, with the ξ_n (or ξ_n*) values of the following line. If the whole procedure were justified we should expect an approximate agreement between these figures. The radius at the end of each time-interval should by this process come out more or less equal to the radius at the beginning of the following interval.

But Table III reveals that the figures in successive lines show hardly any connection at all. We recognise the impossibility in the case of Nova Herculis of combining equations (2) and (3). In other words, the meaning of ξ_1 and ξ_2 in the first equation appears to be quite different from those of ξ_1' and ξ_2' in the second. The radii, the ratio of which we can express through the observed temperature and brightness, do not seem to be those which are associated with the observed velocity curves.

The curious behaviour of the ξ_n and ξ_{n+1} values (and of the ξ_n* and ξ_{n+1}* values) respectively—*i.e.* their decrease with time—is caused according to equation (4) by the corresponding behaviour of $b_{n,\,n+1}$ and $b_{n,\,n+1}$* respectively, since the quantity a does not show big variations, so that the radius essentially varies proportionally to b. The decrease of b in its turn, however,

* S. Chandrasekhar, *M.N.*, **94**, 522, 1934.

† E. G. Williams, *M.N.*, **95**, 579, 1935; improved value see below, § 11.

is caused by the characteristic shape of the velocity curve, as shown in fig. 1, which converges towards the time-axis.

In the case of Nova Aql 1918 the velocity curve before light maximum may have shown a similar appearance; * for Nova Gem 1912 the details are given in Stratton's monograph.†

8. Is the law of variation of the radius with time indeed a different one for equations (2) and (3)?

To answer first the criticism, that the steps used in § 7 might be too short, and therefore the effect of individual fluctuations might be of a very disturbing influence, a similar investigation was made, based on longer, partly overlapping time-intervals, the first stretching from t_1 = December 13·27 to t_{n+1} = December 14·0, the following from 13·27 to 15·0, 13·27 to 16·0, etc. . . . 13·27 to t_{n+1} = December 21·77.

We assume for a moment that both equations at a certain time τ deal really with the same radius, *i.e.* $(\xi)_\tau = (\xi')_\tau$. We choose conveniently τ = December 13·27, being the time of the first velocity observation. Now we take a second date, τ' = December 21·77, being for convenience that of the last observation in Table I, and demand the corresponding agreement:

$$(\xi)_{\tau'} = (\xi')_{\tau'}.$$

If we calculate then those initial values ξ_1 which fulfil these conditions (using the data of Table I) we obtain

$$\xi_1(H) = 58{\cdot}9\odot \quad \text{and} \quad \xi_1(Fe\text{II, etc.}) = 39{\cdot}6\odot.$$

Equation (2) and Table I furnish a series of values $\frac{\xi_{n+1}}{\xi_1}$; using for ξ_1 the values just mentioned for H and *Fe*II, etc. we obtain the curves I and III of fig. 2. On the other hand, we calculate from equation (3) and from the integrations in fig. 1 the corresponding values of $(\xi_{n+1} - \xi_1)$; combining these also with our ξ_1 values we obtain the curves II and IV.

The resulting curves of fig. 2—which thus represent the radii reached at any given moment as function of the time alone—explain fully the reason for the discrepancy found. Curves I and III are of quite a different character from II and IV, the shape of the first two being concave towards the time-axis, that of the others convex. Owing to this shape the values of I with II, or III with IV respectively, cannot be made to agree at any intervening points—*i.e.* at no other moments except the two assumed ones.

9. Recognising thus the impossibility of combining the two kinds of observations involved in equations (2) and (3) to get information about the development of the radius, we may postulate some tentative ideas about the structure of the nova.

We may assume that we have to deal with two different layers in these stars: the one representing the "star" itself is responsible for T and m;

* P. Guthnick, *A.N.*, **207**, 204, 1918; see D. B. McLaughlin, *A.J.*, **45**, 148, 1936.

† F. J. M. Stratton, *Annals, Solar Physics Observatory, Cambridge*, **4**, pt. 1, p. 47, 1920.

the other, the "shell," yields V. Independent of any picture of the mechanism of a nova outburst, we may call these layers the "active surfaces,"

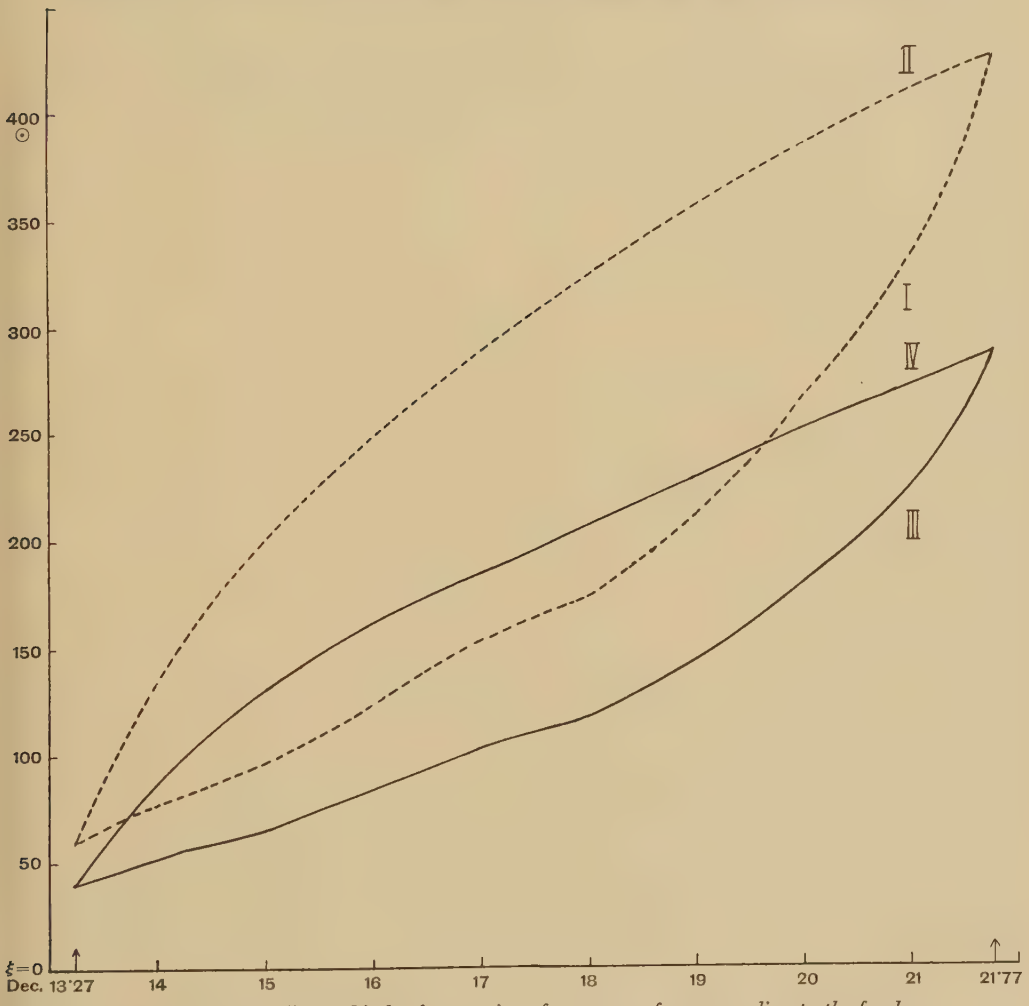


FIG. 2.—*The different kinds of expansion of a nova surface, according to the fundamental equations* (2) *and* (3). *Curves I and III are derived from* (2), *based on brightness and temperature; curves II and IV from* (3), *based on the radial velocity.* (——=*Fe*II *etc.*, – – –=*Hydrogen.*)

and replace the expression star by s, and shell by S, and the radii ξ and ξ' by r and R respectively. Then for the nth interval the equations (2) and (3) take the form :

$$r_{n+1} = r_n \cdot 10^{a_n}, \tag{5}$$

$$R_{n+1} = R_n + b_n, \tag{6}$$

where

$$a_n = \frac{29500(T_n - T_{n+1}) + T_n \cdot T_{n+1}(m_n - m_{n+1})}{5 \cdot T_n \cdot T_{n+1}}, \tag{7}$$

$$b_n = \int_n^{n+1} V \cdot dt. \tag{8}$$

In this connection the recent interesting work by W. Baade and F. Zwicky on super-novæ should be recalled,* which made it "advantageous to visualize two critical surfaces in a super-nova." † Firstly, a "boundary surface, Σ_b," as defined by the outermost layers ejected from the super-nova; and, secondly, a "surface of separation, Σ_s," being the demarcation between

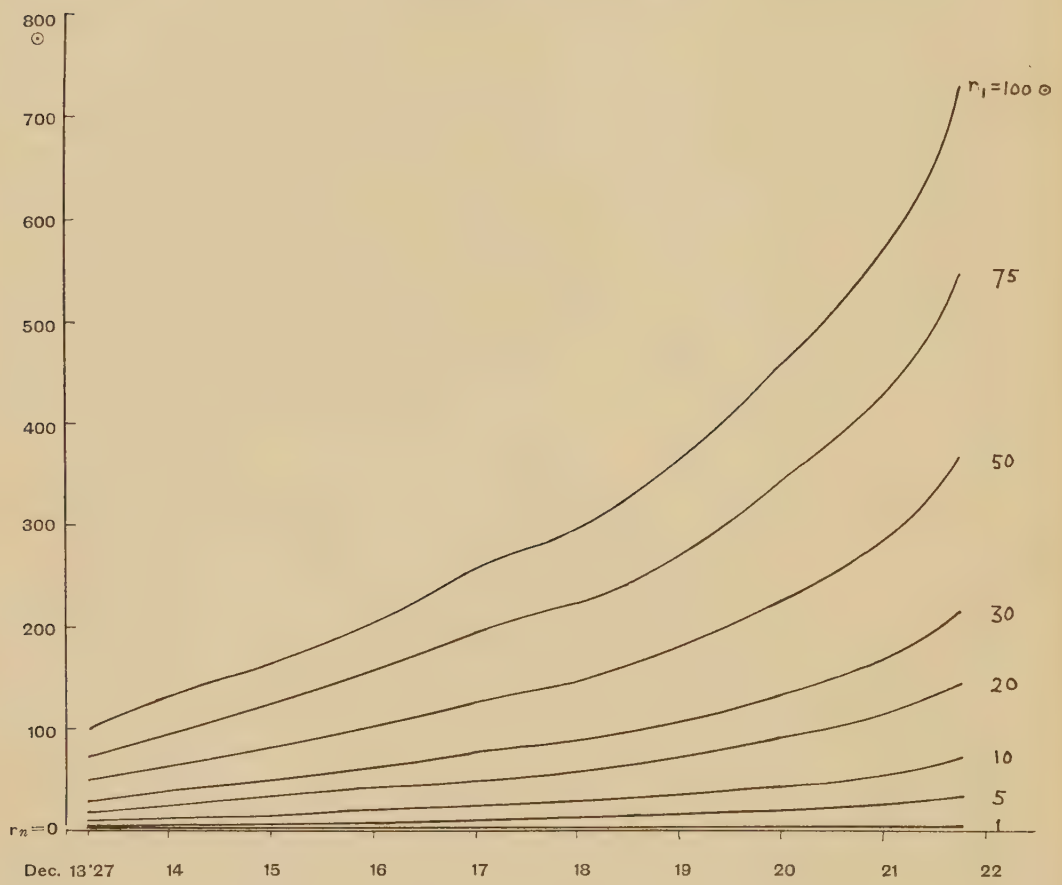


FIG. 3.—*The growth of the inner active surface, s, for different initial radii* r_1.

the gases which are ejected into interstellar space and that matter which remains with the central star. Also, in the case of the Cepheid problem in an investigation recently published by B. W. Kukarkin, ‡ the author feels compelled to consider in a similar way two different layers of the stars, in order to study relations between the different phenomena of stellar variability.

10. Considering N adjacent intervals of time, we see from equations (5) and (6) that we have to deal with $2N$ equations with $2N+2$ unknowns.

* W. Baade, F. Zwicky, *Proc. Nat. Acad.*, **20**, 254 and 259, 1934; *Phys. Rev.*, **46**, 76, 1934; *Proc. Nat. Acad.*, **22**, 457, 1936; *P.A.S.P.*, **48**, 191, 1936.

† F. Zwicky, *Proc. Nat. Acad.*, **22**, 557, 1936.

‡ B. W. Kukarkin, *Russian Astron. Journ.*, **13**, 103, 1936.

For $n = 1$, representing the first interval from the time of first observation, we have

$$r_2 = 10^a \cdot r_1 \quad \text{and} \quad R_2 = R_1 + b. \tag{9}$$

Fig. 3 illustrates the general behaviour of r_n for different initial values of r_1; it is based only on the data of Table I. Similarly, fig. 4 shows the run of the R_n values calculated in using the integrals b of Table III.

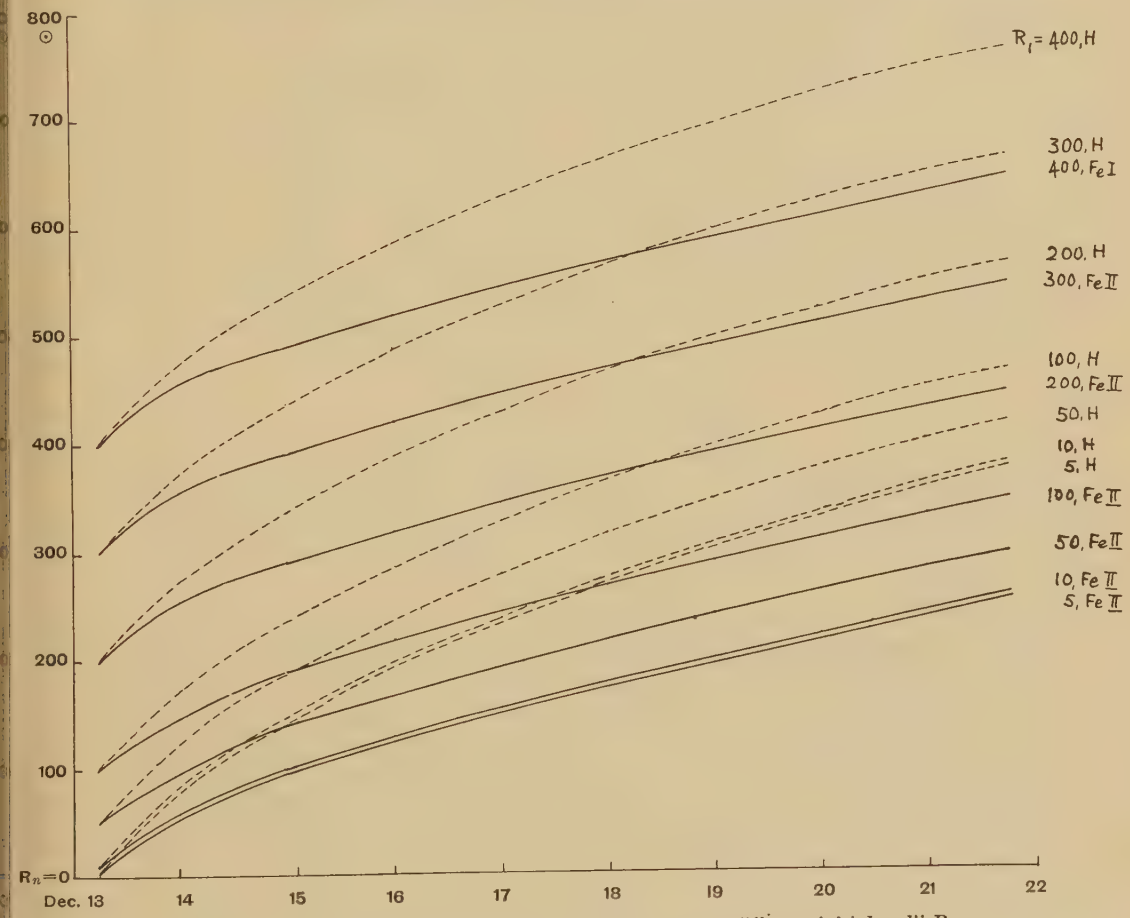


FIG. 4.—*The growth of the outer active surface, s, for different initial radii* R_1.

The curves represent the growth of s and S with time. The surface s is thus equipped with a "constructed velocity curve," which alone fulfils the required conditions of agreement between the results for adjacent time-intervals, and which alone yields to consistent results for the dimensions.

11. The results of the preceding section describe, independently of any definite theory about outburst and structure of the nova, the evolution of the star and the behaviour of its "active surfaces" as soon as we know the two quantities r_1 and R_1 involved in the initial conditions. In order to specify these as reasonably as possible, and then to be able to derive all the

subsequent values r_n and R_n, we try first to fix the date at which the actual outburst of the nova occurred.

We make use of the negative observation of December 12·88 by W. F. A. Ellison (who 7·5 hours before discovery was observing this region of the sky with the naked eye without seeing anything exceptional),* and assume that the magnitude was then $5^m.0$. With a pre-outburst brightness of $14^m.5$, and of $3^m.4$ at discovery, December 13·19,† we get as daily rate of increase $3^m.8$ and a date of outburst $2^d.81$ before discovery. This last figure reads for the other novæ of similar type as follows :—

N. Aur 1892 : $< 2^d.77$,	N. Per 1901 : $1^d.50$,
N. Gem 1912 : $1^d.39$,	N. Aql 1918 ; $0^d.76$,

as deduced from the observations listed by F. J. M. Stratton.‡ With the mean $1^d.85$ of all five cases—keeping of course all the uncertainty in mind—we can assume as the most probable date of outburst : $t^* = 1934$ December 11·34. This is $10^d.85$ before the principal maximum.

The parallax of Nova Herculis was given by E. G. Williams § as $0''.0027$ from his measures of the intensity K of the interstellar calcium line. This value can now be improved by the new general relation :

$$M = m - 12.2 - 5 \log K, \qquad (10)$$

which Williams || published recently, based on a more detailed discussion of his material. For Nova Herculis at maximum we get $M = -7^m.2$, corresponding to $p = 0''.0020$, or to 500 parsecs distance. C. S. Beals,¶ using five plates taken on one night, obtained intensities which give $M = -7^m.1$.

The total increase of brightness is $13^m.2$; hence the absolute magnitude before outburst was $+6^m.0$, which corresponds to $dK2$ as spectral type,** if, as we shall assume, the star was on the main sequence.

Given this type, the radius of the star at this time comes out as follows : $0.83\odot$ from a large material of stars with known parallax †† ; or $1.0\odot$ from equation (1) with $T = 4850°$. From quite different considerations S. Gaposchkin ‡‡ finds $1.46\odot$ as the initial radius of Nova Herculis. We may use a mean value of $1.1\odot$.

12. In order to calculate the dimensions of the star at the time of

* *M.N.*, **96**, 373, 1936.

† This magnitude estimate, due to J. P. M. Prentice, deviates appreciably from the value $3^m.85$ used above in Table I, where the brightness at discovery had to be taken from the smoothed curve, while Prentice's observation obviously coincides with one of the star's rapid fluctuations.

‡ F. J. M. Stratton, *Handbuch d. Astrophysik*, VI, 264, 1928.

§ E. G. Williams, *M.N.*, **95**, 579, 1935.

|| E. G. Williams, *Ap. J.*, **83**, 331, 1936.

¶ C. S. Beals, *M.N.*, **96**, 665, 1936.

** W. S. Adams, A. H. Joy, M. L. Humason, A. M. Brayton, *Ap. J.*, **81**, 196, 1935.

†† *Handbuch d. Astrophysik*, V, 587, 1933.

‡‡ S. Gaposchkin, *Harvard Bull.*, No. 899, p. 8, 1935.

discovery we apply equation (1), taking $M_{13\cdot27} = -4^{m}\cdot6$ and $T = 15{,}000$, and obtain $r_1' = 20\cdot6\odot$.

In the $1^{d}\cdot85$ immediately prior to discovery Nova Herculis therefore increased its radius by $\Delta\xi = 19\cdot5\odot$. This enormous expansion of the star's photosphere s may have led in some way to the expulsion of the matter, which formed the second active surface S, for which in the succeeding days the measured V gave the rate of expansion. The actual separation of the two surfaces we assume to have occurred at a date t_0, which was certainly before discovery because already during the first period of observation the light-increase had slowed down to a comparatively small rate, while the velocity of expansion was still very considerable.

We shall consider three separate cases as follows :—

$$\begin{aligned} t_0' &= t^* + 0\cdot25(t_1 - t^*) = \text{Dec. } 11\cdot80, \\ t_0'' &= t^* + 0\cdot50(t_1 - t^*) = \text{Dec. } 12\cdot26, \\ t_0''' &= t^* + 0\cdot75(t_1 - t^*) = \text{Dec. } 12\cdot72. \end{aligned} \tag{11}$$

The radius ξ_0 on the date when the surface S was cast off can therefore be assumed, approximately enough, as

$$\begin{aligned} \xi_0' &= r_1 + 0\cdot25 \,.\, \Delta\xi = \ 6\cdot0\odot, \\ \xi_0'' &= r_1 + 0\cdot50 \,.\, \Delta\xi = 10\cdot8\odot, \\ \xi_0''' &= r_1 + 0\cdot75 \,.\, \Delta\xi = 15\cdot7\odot. \end{aligned} \tag{12}$$

At these dates we still have $\xi_0 = r_0 = R_0$. The radius R_1 of the surface S at the time t_1 of the first spectroscopic observation, December 13·27, has been calculated from the relation

$$R_1 = R_0 + \int_{t_0}^{t_1} V \,.\, dt, \tag{13}$$

for which purpose it was necessary to extrapolate the observed velocity curve.

The distance, $D = R_1 - r_1$, by which the two surfaces were then separated is, *e.g.*, for t_0''' : $D = 77\cdot4\odot$ for hydrogen, and $D = 48\cdot1\odot$ for *Fe*II, etc.

We are now in a position to solve equations (5) and (6) completely, taking into account the small changes between discovery, December 13·19, and first velocity observation, December 13·27. For this last date we obtain $r_1 = 21\cdot4\odot$.

For the three cases t_0', t_0'', t_0''' we get $R_1 = 257\cdot0$, 150·9, $69\cdot5\odot$ for *Fe*II, etc., and $R_1 = 312\cdot1$, 192·0, 98·8 for hydrogen.

With these initial values the following Table IV has been computed. The second column gives r_n, while the subsequent columns contain the R_n values.

Fig. 5 shows the resulting situation, giving the growth of the active surfaces from outburst up to maximum.

13. Our velocities hitherto used were only "mean velocities," V, given by the displacements of the centres of the absorption lines. To face difficulties which may arise from this fact, it seems advisable to replace V by the maximum and minimum velocities within the lines in question. Let, therefore, V_1 and V_2 represent the velocities at the inner and outer

TABLE IV

Date	r_n	$R_n(t_0')$		$R_n(t_0'')$		$R_n(t_0''')$	
		H	*Fe*II, etc.	*H*	*Fe*II, etc.	*H*	*Fe*II, etc.
Dec. 13·27	21·4	312	257	192	151	99	70
14·0	28·2	385	304	265	198	172	117
15·0	35·3	451	347	331	241	238	160
16·0	44·7	501	377	381	271	288	190
17·0	55·6	542	402	422	296	329	214
18·0	63·3	579	425	459	319	365	237
19·0	77·0	611	447	491	341	398	259
20·0	97·4	639	468	519	362	426	281
21·0	120·7	664	489	544	383	450	302
21·77	155·8	681	506	561	399	468	318

"boundary surfaces" of that region of space which contains the atoms under consideration.

The calculation of the V_1 and V_2 has been performed for all lines, for which the width has been measured, corresponding to the displacements

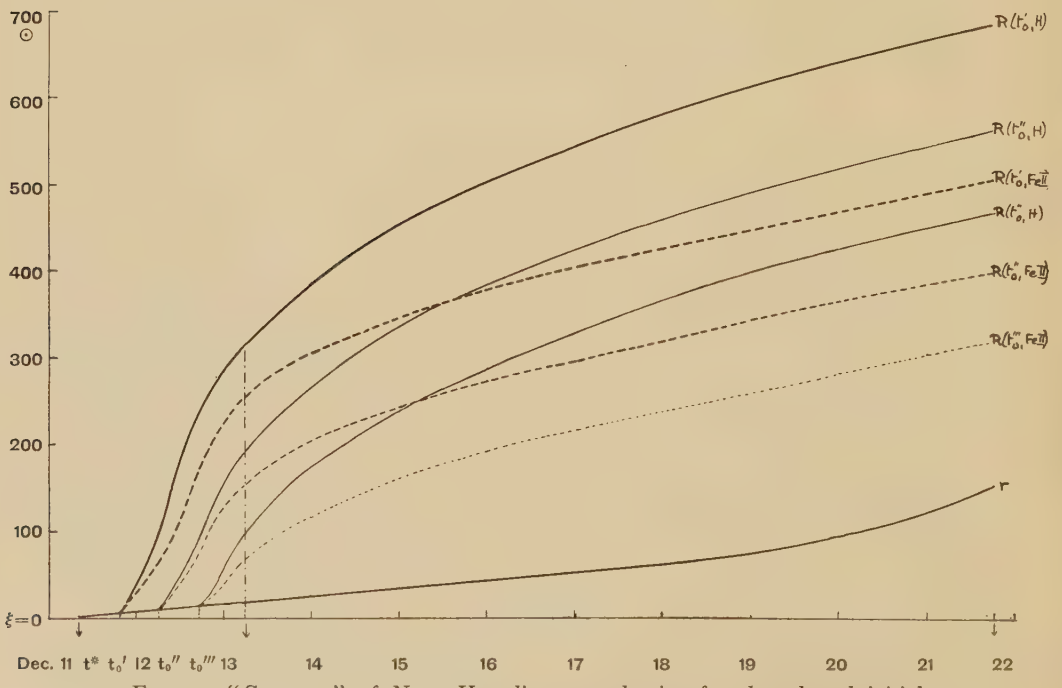


FIG. 5.—*"Structure" of Nova Herculis* 1934, *showing for the adopted initial conditions the radii of inner and outer surfaces. Different curves are given for hydrogen and for Fe*II, *etc., and for three cases of differently assumed dates when the inner and outer surfaces separated.*

of the red and violet edges of the absorption. Only unblended lines were used, and only the range between $H\zeta$ and $H\beta$, which is common to all plates, and therefore alone permits comparison. For the first nova plate taken at the Royal Observatory, Greenwich, the limits of the absorptions were obtained by combining the second and the third columns of Table I, p. 134, in the Cambridge publication.* For the two following Greenwich plates, and for the Cambridge plate of December 17·29, Professor Stratton kindly supplied unpublished measures.

The comparison of the velocities for December 14·28 (Greenwich) and December 14·27 (Cambridge) gave corrections, which have been applied to the velocities from the slitless spectra of December 13·27, 13·77, 14·28; in these cases the values V_1 and V_2 were obtained as a first approximation by using the intervals: centre of emission minus centre of absorption. These corrections, for errors probably caused by the halation of the bright bands spreading into the absorption lines, amount to −20 per cent. at $H\gamma$, to −32 per cent. at λ4481 *Mg*II and to −39 per cent. at $H\beta$. A mean of −30 per cent. was used throughout.

The general run of the figures obtained (the reproduction of which is unnecessary), as well as a graphical analysis, shows no peculiarities which call for a further discussion of the view in mind. The lines of all elements narrow in about the same way as their velocity decreases. It may be of interest to note that the *Mg*II atoms appear to be the slowest ones; further, that the curve for $\frac{1}{2}(V_1+V_2)$ for *Fe*II alone approaches more closely to the H curve than the mean curve for *Fe*II, etc. does in fig. 1. Concluding, we may say that "both edges of the shell" behave for different atoms in the same general manner as outlined in the present discussion.

The picture developed in this paper works with the assumption that the atoms of all elements leave the star together and at the same moment. Their initial velocities are different, and are slowed down at a different rate too. At the end of the first phase the hydrogen atoms would have reached, as we have seen—if this assumption is valid—a distance from the star about 50 per cent. greater than that of the *Fe*II and other atoms.†

But, on the other hand, we notice that at this time all the velocities have slowed down to just the same speed, −170 km./sec. That this effect would have been reached in the same way in quite different layers round the star does not seem probable. Rather we would feel inclined to think that at this stage all the atoms had united to a common assemblage of ejected matter, and that it is for this reason that they exhibit the same velocity.

There is indeed no necessity to assume that *Fe*II, etc. had started at the same time as hydrogen. It was not until December 14·27 that it was first observed in the spectrum. But if it had started later than hydrogen the final difference between our two distances would have become greater still.

* F. J. M. Stratton, *Annals, Solar Physics Observatory, Cambridge*, 4, pt. 4.

† This alone shows the impossibility of securing agreement of *both* of these radii with the radius otherwise deduced, and points at once to the difficulties underlying the hypothesis criticised in this paper. This point is to be emphasised since it is independent of any evaluation of temperature.

Probably only the high temperature in the very first stage prevented its earlier appearance in the spectrum.

The assumption that the origin of the ejected hydrogen lay deeper than that of iron, so that the latter started earlier, appears to be one way out, but certainly not a very probable one.

We may also consider the idea of an initially resisting medium * in the neighbourhood of the star, existing only for a short time after the outburst. As long as the ejected atoms are in counteraction against this medium, all our velocity results will be influenced by its presence, and obviously the more, the lighter the atoms involved.

When by the initial process of expansion—explosion or the like—the atoms were ejected, the initial speed which hydrogen received may have been, by some reasons of atomic constitution, a much bigger one than for the other elements. But the resistance acted much more strongly on these light atoms, and their slowing down, in consequence, was a faster one than for *Fe*II, etc. When *H* had slowed down to the same velocity as *Fe*II, etc. the velocity curve shows that all the elements had already settled down to about the same velocity.† It is obvious that this rough picture at least accounts for the shape of the observed velocity curves for the different elements.

Whatever hypothesis we make, however, hydrogen is more likely to be affected by outside factors than the other atoms.

The idea of the existence of two active surfaces *s* and *S* is, of course, only regarded tentatively. The results based on it, and shown especially in fig. 5, are to be supplemented by the above considerations.

14. *Summary.*—(1) The attempt to combine temperature *T*, brightness *m* and absorption line radial velocities *V*, as observed in some novæ in the phase before maximum, leads to fundamental discrepancies.

(2) The introduction of two active surfaces *s* and *S* with the radii *r* and *R* is put forward as a tentative suggestion, which removes the difficulties —the expansion of the one being responsible for the changes in *T* and *m*, the expansion of the other for the observed *V* curve.

(3) Equations for the rate of increase of *r* and *R* are derived and the results for different assumed initial conditions are given.

(4) The most probable assumptions lead to a date of the outburst of Nova Herculis 1934 of December 11·34, $1^{d}\cdot85$ before discovery. If three-quarters of the way through this first development, from outburst to the first observation, the outer active surface *S* was cast off from the inner *s* (the expansion velocities and the radii of both surfaces being then identical), we find for the radius of the inner "star" at the time of discovery 21⊙, and for the distance of the second active surface from this inner one : 77⊙ for hydrogen, and 48⊙ for *Fe*II, etc.

(5) With these initial values the quantities *r* and *R* are represented as

* F. J. M. Stratton, *The Observatory*, 59, 330, 1936.

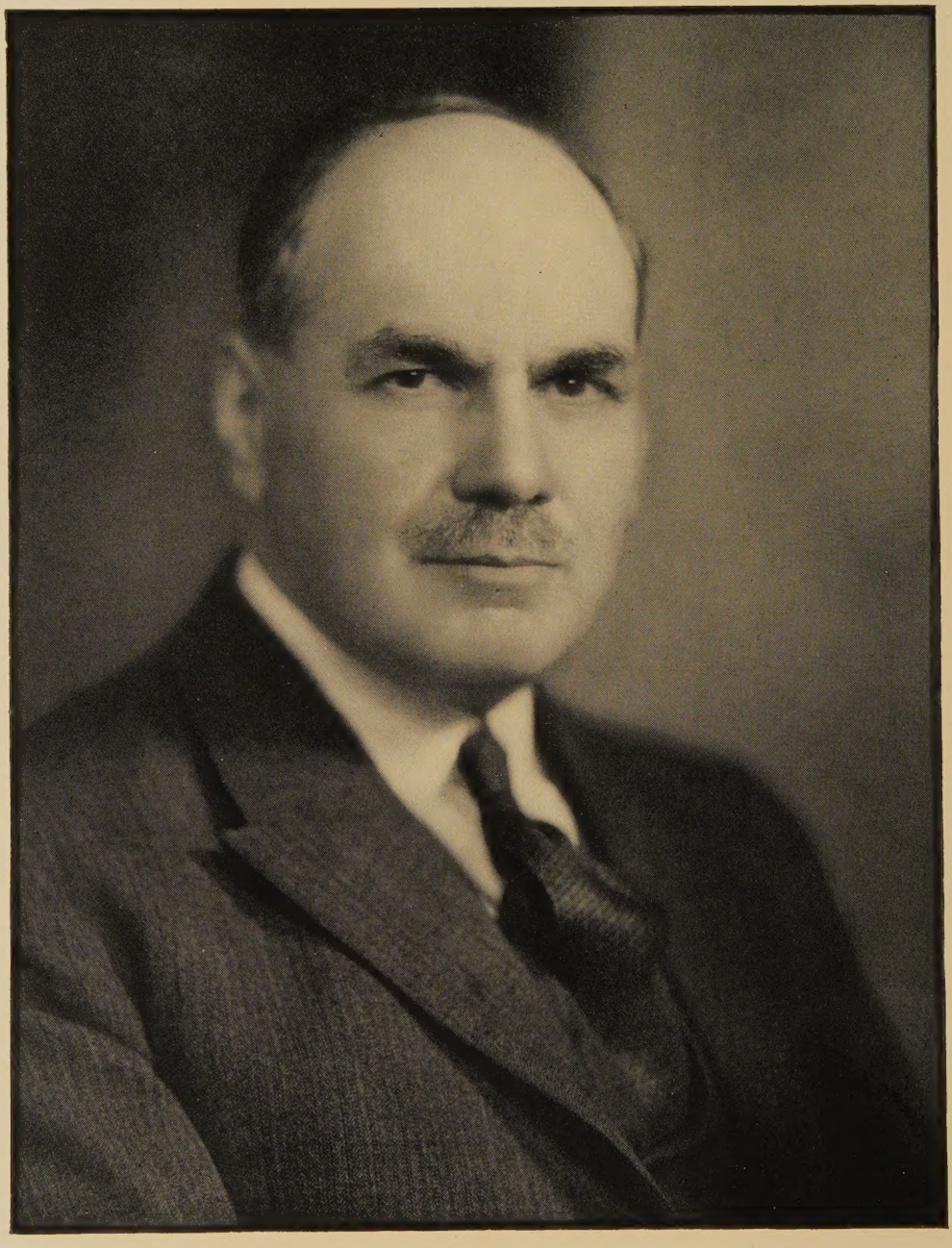
† It is perhaps significant that this coincidence of the different velocity curves occurs just at the time of light maximum.

functions of time. It is, at a time shortly before light maximum (for the initial quantities just given), $r = 156\odot$; $R = 561\odot$ for hydrogen and $399\odot$ for *Fe*II, etc.

(6) The behaviour of the different atoms is discussed with regard to possible pictures of the nova outburst.

I have great pleasure in expressing my sincere thanks to Professor F. J. M. Stratton and to Mr. E. G. Williams, for helpful criticism and stimulating discussions.

Solar Physics Observatory,
Cambridge :
1937 *January* 2.



MR J. H. REYNOLDS, M.SC.
President, 1935–37.